Study of Best Proximity Points Results for Some Class of Multivalued Contractions

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of degree

Master of Science

in

Mathematics

by

Mohit

(MS23MTH014)

under the guidance of

Dr. Deepesh Kumar Patel

Department of Mathematics

Visvesvaraya National Institute of Technology

Nagpur 440010 (India)

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Declaration

I, Mohit, thus declare that this dissertation titled "**Study of Best Proximity Points Results for Some Class of Multivalued Contractions**" is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is unique and has never been presented before, in whole or in part, for any degree or diploma in this or any other institution.

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This is to certify that the dissertation titled "**Study of Best Proximity Points Results for Some Class of Multivalued Contractions**" is submitted by **Mohit** in partial fulfillment of the requirements for the award of the degree of **Master of Science in Mathematics**, VNIT Nagpur. The work is thorough complete and ready for final assessment.

Dr. Deepesh Kumar Patel
Assistant Professor, Mathematics
VNIT, Nagpur

Head, Department of Mathematics
VNIT, Nagpur
Date:

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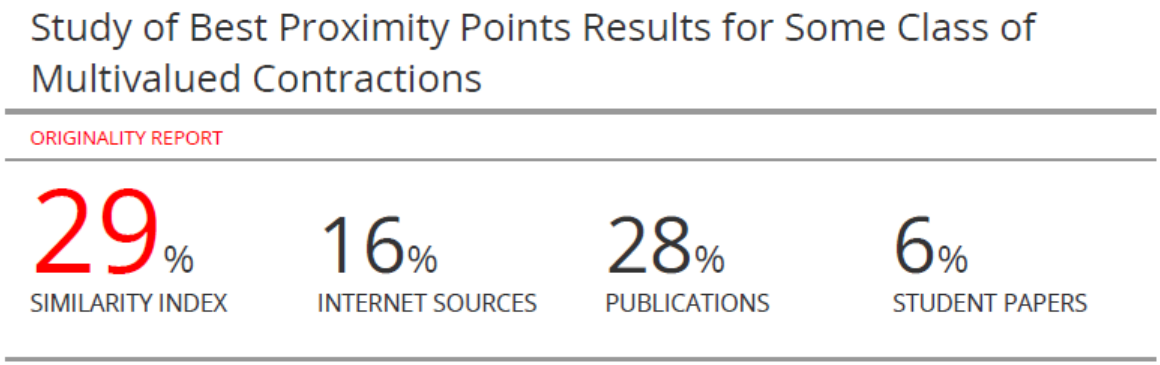

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Name of the student : MOHIT

Enrollment No. : MS23MTH014

ID No. 30379

Dr. Deepesh Kumar Patel

(Supervisor)

Assistant Professor, Mathematics

VNIT, Nagpur

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Mohit

Abstract

In classical fixed point theory, the condition $\varkappa = \Gamma\varkappa$ assumes that the mapping is self-mapping, which often fails in real-world scenarios such as optimization and approximation problems. The best proximity point theory overcomes this limitation by seeking approximate solutions that minimize the distance between a point and its image, particularly when the mapping does not admit fixed points. This thesis focuses on the study of best proximity point results for multivalued mappings and their applications in applied mathematics, particularly in homotopy theory. The structure of the thesis is organized as follows: Chapter 1 introduces the necessary notations, basic definitions, and preliminary concepts, along with some fixed point theorems for multivalued mappings. Chapter 2 presents sufficient conditions for the existence of best proximity points for non-self multivalued contractions. These results generalize the Nadler contraction principle and the Feng-Liu fixed point theorem. In Chapter 3, we extend the study to best proximity point theorems for cyclic mappings, including cyclic contractions and (U, V)-Feng-Liu type cyclic contraction pairs. Chapter 4 explores the existence of best proximity points using multivalued F-contractions, also known as Wardowski's technique, which generalizes classical contraction principles and provides a broader framework for analyzing non-self mappings. Chapter 5 investigates best proximity point results in abstract spaces, specifically in 0-complete partial metric spaces, along with applications in homotopy theory. Finally, Chapter 6 presents the conclusions and outlines directions for future research.

This report is based on the study of following book and research articles:

- Agarwal, R. P., O'Regan, D., and Sahu, D. R. Fixed point theory for Lipschitzian-type mappings with applications Vol. 6. New York: Springer, 2009.

- Abkar, A., and Gabeleh, M. The existence of best proximity points for multivalued non-self-mappings. RACSAM, 107(2) (2013), 319-325.

- Sahin, H., Aslantas, M., and Altun, I. Feng-Liu type approach to best proximity point results for multivalued mappings. Journal of Fixed Point Theory and Applications, 22(1) (2020), Article ID 11.

- Eldred, A. A., and Veeramani, P. Existence and convergence of best proximity points. Journal of Mathematical Analysis and Applications, 323(2) (2006), 1001-1006.

- Debnath, P. Optimization through best proximity points for multivalued F-contractions. Miskolc Mathematical Notes, 22(1) (2021), 143-151.

- Aslantas, M. Finding a solution to an optimization problem and an application. Journal of Optimization Theory and Applications, 194(1) (2022), 121-141.

List of Notations

$\mathbb{C}$	: Complex numbers' set
$\mathbb{R}$	: Real numbers' set
$\mathbb{N}$	: Natural numbers' set
$\rightarrow$	: maps to
$\subset$	: subset of
$\Gamma : \mathcal{Y} \rightarrow \mathcal{Y}$	: Γ is a mapping from $\mathcal{Y}$ into $\mathcal{Y}$
$\Gamma(\varkappa)$ or $\Gamma\varkappa$	: Image of $\varkappa$ under Γ
iff	: If and only if
$\max$	: Maximum
$\min$	: Minimum
$\inf$	: Infimum
$\varnothing$	: Empty set
$\in$	: Belongs to
$\exists$	: There exists
$(\mathcal{Y}, d)$	: A metric space with metric d
$\mathcal{H}(U, V)$	: Housdorff distance between two sets $U \& V$
$\mathrm{dist}(U, V)$	: Distance between two sets U and V
$\forall$	: For all

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Chapter 1

Introduction

Fixed point theory is a central topic in mathematical analysis, with broad applications in areas such as differential equations, optimization, and computer science. A fixed point of a function is a point that is mapped to itself by the function. More formally, given a function Γ defined on a set $\mathcal{Y}$ a point $\varkappa \in \mathcal{Y}$ is known as a fixed point of Γ if $\Gamma(\varkappa) = \varkappa$. The concept, while simple in definition, plays a central role in various branches of mathematics such as topology, functional analysis, and differential equations. Moreover, fixed point results form the theoretical backbone of many applied fields, including economics, game theory, computer science, and engineering.

One of the foundational results in this area is the Banach fixed point theorem. This theorem provides sufficient conditions under which a function admits a unique fixed point and guarantees the convergence of an iterative process to that point. Suppose given a metric space $(\mathcal{Y}, \rho)$ and let $\Gamma : \mathcal{Y} \to \mathcal{Y}$ be a contraction mapping, i.e. $\exists$ a constant $0 < c < 1$ such that

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) \leq c\,\rho(\varkappa, \mathfrak{y}) \;\forall\; \varkappa, \mathfrak{y} \in \mathcal{Y},$$

then Γ possesses a unique fixed point in $\mathcal{Y}$. Furthermore, for any initial value $\varkappa_0 \in \mathcal{Y}$, the sequence $\{\Gamma^n \varkappa_0\}$ converges to this fixed point. The significance of Banach's theorem lies in its constructive nature, offering not only existence and uniqueness of fixed points but also a practical method for approximating them. This result has inspired a vast body of research extending the theorem to more general spaces, weaker contractive conditions, and multivalued mappings.

In this chapter, we study the extension of Banach contraction principle, particularly focusing on fixed point theorems for multivalued mappings in metric.

1.1 Multivalued mappings

Let $U \neq \varnothing$ be a subset of a metric space $(\mathcal{Y}, \rho)$. Define

$$\rho(\varkappa, U) = \inf\{\rho(\varkappa, \mathfrak{y}) : \mathfrak{y} \in U\}.$$

Following are the notations for different kinds of families of sets:

- The notation $\mathcal{N}(\mathcal{Y})$ refers to the collection of all non-empty subsets of the space $\mathcal{Y}$,
- The symbol $\mathcal{C}(\mathcal{Y})$ is used to represent the collection of non-empty, closed subsets of $\mathcal{Y}$,
- The set $\mathcal{CB}(\mathcal{Y})$ includes all subsets of $\mathcal{Y}$ that are simultaneously non-empty, closed, and bounded,
- The notation $\mathcal{K}(\mathcal{Y})$ stands for the family of subsets of $\mathcal{Y}$ that are both non-empty and compact.

Remark 1.1.1. *It is evident that each compact subset of $\mathcal{Y}$ must also be closed and bounded, which implies the inclusion $\mathcal{K}(\mathcal{Y}) \subseteq \mathcal{CB}(\mathcal{Y})$.*

Definition 1.1.1 (Hausdorff distance)**.** *For $U, V \in \mathcal{CB}(\mathcal{Y})$, we define the Hausdorff distance as*

$$\mathcal{H}(U, V) = \max\{\Delta(U, V), \Delta(V, U)\} = \max\left\{\sup_{\varkappa \in U} \rho(\varkappa, V), \sup_{\mathfrak{y} \in V} \rho(\mathfrak{y}, U)\right\}.$$

where, $\Delta(U, V) = \sup\{\rho(\varkappa, V) : \varkappa \in U\}$.
***Note:** "Δ is not necessarily symmetric".*

Example 1.1.1. *Let $U = [0, 2]$ and $V = \{3\}$, then*

$$\Delta(U, V) = \sup_{\varkappa \in U} \rho(\varkappa, 3) = \sup_{\varkappa \in [0,1]} |\varkappa - 3| = 3 \text{ and}$$
$$\Delta(V, U) = \sup_{\varkappa \in V} \rho(\varkappa, U) = \rho(3, U) = \inf_{\varkappa \in U} \rho(3, \varkappa) = \inf_{\varkappa \in [0,1]} |3 - \varkappa| = 1.$$

Therefore, $\Delta(U, V) \neq \Delta(V, U)$, i.e. Δ is not symmetric.
Also, $\mathcal{H}(U, V) = \max\{\Delta(U, V), \Delta(V, U)\} = 3$.

Example 1.1.2. *Let* $\mathcal{Y} = \mathbb{R},\ U = [1, 2]$ *and* $V = [2, 3]$. *Then*

$$\Delta(U, V) = \sup_{\varkappa \in U} \rho(\varkappa, V) = \sup_{\varkappa \in [1,2]} |\varkappa - 2| = 1 \text{ and}$$
$$\Delta(V, U) = \sup_{\varkappa \in V} \rho(\varkappa, U) = \sup_{\varkappa \in [2,3]} |\varkappa - 1| = 1.$$

Here $\Delta(U, V) = \Delta(V, U) = 1$ *and hence* $\mathcal{H}(U, V) = 1$.

Proposition 1.1.1. *Let* $(\mathcal{Y}, \rho)$ *be a metric space, and let* $U, V, W \in \mathcal{CB}(\mathcal{Y})$. *Then*

(i) $\Delta(U, V) = 0 \iff U \subset V$,

(ii) $V \subset W \Rightarrow \Delta(U, W) \leq \Delta(U, V)$,

(iii) $\Delta(U \cup V, W) = \max\{\Delta(U, W), \Delta(V, W)\}$,

(iv) $\Delta(U, V) \leq \Delta(U, W) + \Delta(W, V)$.

Proof. (i) By definition of Δ, we have

$$\Delta(U, V) = 0 \iff \sup\{\rho(\varkappa, V) : \varkappa \in U\} = 0$$
$$\iff \rho(\varkappa, V) = 0,\ \forall \varkappa \in U.$$

Because V is closed in $\mathcal{Y}$, we have $\rho(\varkappa, V) = 0 \iff \varkappa \in V$.

Thus,

$$\Delta(U, V) = 0 \iff U \subset V.$$

(ii) Observe that,

$$V \subset W \Rightarrow \rho(\varkappa, W) \leq \rho(\varkappa, V) \quad \forall \varkappa \in \mathcal{Y}$$
$$\Delta(U, W) \leq \Delta(U, V).$$

(iii) To show $\Delta(U \cup V, W) = \max\{\Delta(U, W), \Delta(V, W)\}$. Observe that,

$$\Delta(U \cup V, W) = \sup_{\varkappa \in U \cup V} \rho(\varkappa, W)$$
$$= \max\{\sup_{\varkappa \in U} \rho(\varkappa, W), \sup_{\varkappa \in V} \rho(\varkappa, W)\}$$
$$= \max\{\Delta(U, W), \Delta(V, W)\}.$$

(iv) Let $\mathfrak{u} \in U, \mathfrak{v} \in V, \mathfrak{w} \in W$. Then

$$\begin{aligned}
&\rho(\mathfrak{u}, \mathfrak{v}) \leq \rho(\mathfrak{u}, \mathfrak{w}) + \rho(\mathfrak{w}, \mathfrak{v}) \\
\Rightarrow\ &\rho(\mathfrak{u}, V) \leq \rho(\mathfrak{u}, \mathfrak{w}) + \rho(\mathfrak{w}, V) \\
\Rightarrow\ &\rho(\mathfrak{u}, V) \leq \rho(\mathfrak{u}, \mathfrak{w}) + \rho(W, V) \\
\Rightarrow\ &\rho(U, V) \leq \rho(U, W) + \rho(W, V) \\
\Rightarrow\ &\Delta(U, V) \leq \Delta(U, W) + \Delta(W, V).
\end{aligned}$$

Proposition 1.1.2. *$\mathcal{H}$ is a metric on $\mathcal{CB}(\mathcal{Y})$ if $(\mathcal{Y}, \rho)$ is a metric space..* □

Proof. By the definition of $\mathcal{H}$, we have $\mathcal{H}(U, V) \geq 0$ and $\mathcal{H}(U, V) = \mathcal{H}(V, U)$. Observe that

$$\begin{aligned}
\mathcal{H}(U, V) = 0 &\iff \max\{\Delta(U, V), \Delta(V, U)\} = 0 \\
&\iff \Delta(U, V) = 0 \quad \text{and} \quad \Delta(V, U) = 0 \\
&\iff U \subset V \quad \text{and} \quad V \subset U \\
&\iff U = V.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{H}(U, V) &= \max\{\Delta(U, V), \Delta(V, U)\} \\
&\leq \max\{\Delta(U, W) + \Delta(W, V), \Delta(V, W) + \Delta(W, U)\} \\
&\leq \max\{\Delta(U, W), \Delta(W, U)\} + \max\{\Delta(V, W), \Delta(W, V)\} \\
&= \mathcal{H}(U, W) + \mathcal{H}(W, V).
\end{aligned}$$

As a result, the function $\mathcal{H}$, known as the Hausdorff metric, defines a valid metric on the space $\mathcal{CB}(\mathcal{Y})$. □

Remark 1.1.2. *Let $U, V \in \mathcal{CB}(\mathcal{Y})$ and $\mathfrak{u} \in U$. Then for $\epsilon > 0$, $\rho(\mathfrak{u}, \mathfrak{v}) \leq \mathcal{H}(U, V) + \epsilon$ for some point $\mathfrak{v} \in V$.*

Proposition 1.1.3. *Suppose given a metric space $(\mathcal{Y}, \rho)$. Then*

$$\mathcal{H}(U \cup V, W \cup Z) \leq \max\{\mathcal{H}(U, W), \mathcal{H}(V, Z)\} \ \ \forall U, V, W, Z \in \mathcal{CB}(\mathcal{Y}).$$

Proof.

$$\begin{aligned}\Delta(U \cup V, W \cup Z) &= \max\{\Delta(U, W \cup Z), \Delta(V, W \cup Z)\} \\ &\leq \max\{\Delta(U, W), \Delta(V, Z)\} \\ &\leq \max\{\Delta(V, W), \Delta(V, Z)\} \\ &= \max\{\mathcal{H}(U, W), \mathcal{H}(V, Z)\}.\end{aligned}$$

Similarly, we have

$$\Delta(W \cup Z, U \cup V) = \max\{\mathcal{H}(U, W), \mathcal{H}(V, Z)\}.$$

By definition of $\mathcal{H}$, we have

$$\begin{aligned}\mathcal{H}(U \cup V, W \cup Z) &= \max\{\Delta(U \cup V, W \cup Z), \Delta(W \cup Z, U \cup V)\} \\ &\leq \max\{\Delta(U, W), \Delta(V, Z)\}, \max\{\Delta(W, U), \Delta(Z, V)\} \\ &= \max\{\mathcal{H}(U, W), \mathcal{H}(V, Z)\}.\end{aligned}$$

□

Definition 1.1.2. *Let $g : \mathcal{Y} \to \mathbb{R}$ be a mapping. Then g is said to be lower semi-continuous if the set*

$$g^{-1}((\beta, \infty)) = \{\varkappa \in \mathcal{Y} : g(\varkappa) > \beta\}$$

is an open set for all $\beta \in \mathbb{R}$. The function g called as lower semi-continuous on $\mathcal{Y}$ if it is lower semi-continuous at each $\varkappa \in \mathcal{Y}$.

Definition 1.1.3. *Let $g : \mathcal{Y} \to \mathbb{R}$ be a mapping. Then g is called as upper semi-continuous if the set*

$$g^{-1}((-\infty, \beta)) = \{\varkappa \in \mathcal{Y} : g(\varkappa) < \beta\}$$

is an open set for all $\beta \in \mathbb{R}$. The function g called as upper semi-continuous on $\mathcal{Y}$ if it is upper semi-continuous at each $\varkappa \in \mathcal{Y}$.

Proposition 1.1.4. *Let $(\mathcal{Y}, \rho)$ be a metric space $(\mathcal{Y}, \rho)$ and $\varnothing \neq U \subset \mathcal{Y}$, and assume that the set-valued mapping $\Gamma : U \to \mathcal{C}(\mathcal{Y})$ is upper semi-continuous at a point $\varkappa_0 \in U$. Then the function*

$\theta : U \to \mathbb{R}^+$, *given by* $\theta(\varkappa) = \rho(\varkappa, \Gamma\varkappa)$, *is lower semi-continuous at* $\varkappa_0$.

Proof. Fix any $\epsilon > 0$. Since Γ is upper semi-continuous at $\varkappa_0$, $\exists$ a $\delta > 0$ such that for all $\mathfrak{y} \in U$ with $\rho(\varkappa_0, \mathfrak{y}) < \delta$, the image $\Gamma\mathfrak{y}$ lies within an $\epsilon/4$-neighborhood of $\Gamma\varkappa_0$. We may further assume $\delta \leq \epsilon/4$ for convenience.
Now, pick any such $\mathfrak{y} \in U$ with $\rho(\varkappa_0, \mathfrak{y}) < \delta$, and select $\mathfrak{u} \in \Gamma\mathfrak{y}$ satisfying

$$\rho(\mathfrak{y}, \mathfrak{u}) \leq \rho(\mathfrak{y}, \Gamma\mathfrak{y}) + \epsilon/2.$$

Because $\Gamma\mathfrak{y}$ is close to $\Gamma\varkappa_0$, we can find $\mathfrak{v} \in \Gamma\varkappa_0$ such that $\rho(\mathfrak{u}, \mathfrak{v}) \leq \epsilon/4$. Then,

$$\begin{aligned}\rho(\varkappa_0, \Gamma\varkappa_0) - [\rho(\mathfrak{y}, \Gamma\mathfrak{y}) + \epsilon/2] &\leq \rho(\varkappa_0, \Gamma\varkappa_0) - \rho(\mathfrak{y}, \mathfrak{u}) \\ &\leq \rho(\varkappa_0, \mathfrak{v}) - \rho(\mathfrak{y}, \mathfrak{u}) \\ &\leq \rho(\varkappa_0, \mathfrak{y}) + \rho(\mathfrak{y}, \mathfrak{u}) + \rho(\mathfrak{u}, \mathfrak{v}) - \rho(\mathfrak{y}, \mathfrak{u}) \\ &= \rho(\varkappa_0, \mathfrak{y}) + \rho(\mathfrak{u}, \mathfrak{v}) \leq \epsilon/2.\end{aligned}$$

Hence,

$$\rho(\varkappa_0, \Gamma\varkappa_0) \leq \rho(\mathfrak{y}, \Gamma\mathfrak{y}) + \epsilon,$$

which shows that $\theta(\varkappa)$ is lower semi-continuous at $\varkappa_0$. □

1.2 Fixed point theorems for multivalued mappings

Definition 1.2.1. *An element* $\varkappa \in \mathcal{Y}$ *is known as a fixed point of a multivalued mapping* $\Gamma : \mathcal{Y} \to \mathcal{N}(\mathcal{Y})$ *if* $\varkappa \in \Gamma(\varkappa)$.

Definition 1.2.2. *Consider a metric space* $(\mathcal{Y}, \rho)$ *and suppose given a set-valued mapping* $\Gamma : \mathcal{Y} \to \mathcal{N}(\mathcal{Y})$. *The mapping* Γ *is called as Lipschitz-type if* $\exists$ *a constant* $\lambda > 0$ *such that* $\forall\ \varkappa, \mathfrak{y} \in \mathcal{Y}$, *the inequality*

$$\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y})) \leq \lambda\, \rho(\varkappa, \mathfrak{y})$$

holds, where $\mathcal{H}$ *denotes the Hausdorff distance between the images of the mapping.*

- *The mapping* Γ *is known as contractive multivalued map if* $\lambda < 1$, *and nonexpansive if* $\lambda = 1$.

- *The collection of fixed points of Γ, denoted by $F(\Gamma)$, consists of all points $\varkappa \in \mathcal{Y}$ such that $\varkappa \in \Gamma(\varkappa)$, that is,*

$$F(\Gamma) = \{\varkappa \in \mathcal{Y} \mid \varkappa \in \Gamma(\varkappa)\}.$$

Theorem 1.2.1 (Nadler fixed point theorem)**.** *Consider a contraction mapping $\Gamma : \mathcal{Y} \to \mathcal{CB}(\mathcal{Y})$ and suppose given a complete metric space $(\mathcal{Y}, \rho)$. Then Γ possesses a fixed point in $\mathcal{Y}$.*

Proof. Let $\lambda \in (0, 1)$ and select an initial element $\varkappa_0 \in \mathcal{Y}$, and choose $\varkappa_1 \in \Gamma\varkappa_0$. According to Remark 1.1.2, we can find $\varkappa_2 \in \Gamma\varkappa_1$ such that

$$\rho(\varkappa_1, \varkappa_2) \leq \mathcal{H}(\Gamma\varkappa_0, \Gamma\varkappa_1) + \lambda.$$

Continuing this process, we obtain $\varkappa_3 \in \Gamma\varkappa_2$ satisfying

$$\rho(\varkappa_2, \varkappa_3) \leq \mathcal{H}(\Gamma\varkappa_1, \Gamma\varkappa_2) + \lambda^2.$$

Proceeding inductively, we get a sequence $\{\varkappa_n\}$ in $\mathcal{Y}$ with $\varkappa_{n+1} \in \Gamma\varkappa_n$ and

$$\rho(\varkappa_n, \varkappa_{n+1}) \leq \mathcal{H}(\Gamma\varkappa_{n-1}, \Gamma\varkappa_n) + \lambda^n \quad \text{for all } n \in \mathbb{N}.$$

We now analyze the distance between successive terms:

$$\begin{aligned}
\rho(\varkappa_n, \varkappa_{n+1}) &\leq \mathcal{H}(\Gamma\varkappa_{n-1}, \Gamma\varkappa_n) + \lambda^n \\
&\leq \lambda\rho(\varkappa_{n-1}, \varkappa_n) + \lambda^n \\
&\leq \lambda\left(\lambda\rho(\varkappa_{n-2}, \varkappa_{n-1}) + \lambda^{n-1}\right) + \lambda^n \\
&= \lambda^2\rho(\varkappa_{n-2}, \varkappa_{n-1}) + 2\lambda^n \\
&\;\;\vdots \\
&\leq \lambda^n\rho(\varkappa_0, \varkappa_1) + n\lambda^n.
\end{aligned}$$

As both $\sum_{n=0}^{\infty} \lambda^n$ and $\sum_{n=0}^{\infty} n\lambda^n$ converge for $0 < \lambda < 1$, it follows that:

$$\sum_{n=0}^{\infty} \rho(\varkappa_n, \varkappa_{n+1}) \leq \sum_{n=0}^{\infty} (\lambda^n\rho(\varkappa_0, \varkappa_1) + n\lambda^n)$$

$$= \rho(\varkappa_0, \varkappa_1) \sum_{n=0}^{\infty} \lambda^n + \sum_{n=0}^{\infty} n\lambda^n$$

$$< \infty.$$

This proves that $\{\varkappa_n\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is complete, $\exists\, \mathfrak{v} \in \mathcal{Y}$ such that $\lim_{n\to\infty} \varkappa_n = \mathfrak{v}$.

since Γ is continuous, we have:

$$\lim_{n\to\infty} \mathcal{H}(\Gamma\varkappa_n, \Gamma\mathfrak{v}) = 0.$$

Given that $\varkappa_{n+1} \in \Gamma\varkappa_n$, it follows that:

$$\lim_{n\to\infty} \rho(\varkappa_{n+1}, \Gamma\mathfrak{v}) = 0 \quad \Rightarrow \quad \rho(\mathfrak{v}, \Gamma\mathfrak{v}) = 0.$$

As $\Gamma\mathfrak{v}$ is closed, this implies $\mathfrak{v} \in \Gamma\mathfrak{v}$, completing the proof. □

Example 1.2.1. *Given the set $\mathcal{Y} = [0, 1]$, and consider a function $g : \mathcal{Y} \to \mathbb{R}$ defined by*

$$g(\varkappa) = \begin{cases} \frac{\varkappa}{2} + \frac{1}{2}, & \text{if } 0 \le \varkappa \le \frac{1}{2}, \\ -\frac{\varkappa}{2} + 1, & \text{if } \frac{1}{2} \le \varkappa \le 1. \end{cases}$$

Next, define a mapping $\Gamma : \mathcal{Y} \to 2^{\mathcal{Y}}$ such that $\Gamma(\varkappa) = \{g(\varkappa)\} \cup \{0\}$ for each $\varkappa \in \mathcal{Y}$. Then, Γ satisfies the conditions of a multivalued contraction. It can be verified that 0 and $\frac{2}{3}$ are fixed points of Γ, that is, $F(\Gamma) = \left\{0, \frac{2}{3}\right\}$.

Note 1. *The above example shows that the fixed point of a multivalued contraction may not be unique.*

Theorem 1.2.2 (Feng-Liu fixed point theorem)**.** *Consider a complete metric space $(\mathcal{Y}, d)$ and a multivalued mapping $\Gamma : \mathcal{Y} \to \mathcal{C}(\mathcal{Y})$, where $\mathcal{C}(\mathcal{Y})$ denotes the collection of all closed subsets of $\mathcal{Y}$. Suppose there exists a constant $c \in (0, 1)$ such that for every $\varkappa \in \mathcal{Y}$, one can find a point $\mathfrak{y}$ in the set*

$$\mathcal{I}_b^{\varkappa} := \{\mathfrak{y} \in \Gamma(\varkappa) : b \cdot d(\varkappa, \mathfrak{y}) \le \mathcal{D}(\mathfrak{y}, \Gamma(\varkappa))\},$$

for some $b > 0$, such that

$$\mathcal{D}(\mathfrak{y}, \Gamma(\mathfrak{y})) \le c \cdot d(\varkappa, \mathfrak{y}).$$

If the constant c is strictly less than b, and the function g is lower semi-continuous, then the mapping Γ admits at least one fixed point in $\mathcal{Y}$.

Example 1.2.2. *Let us take the set*

$$\mathcal{Y} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right\} \cup \{0, 1\},$$

and the standard metric

$$d(\varkappa, \mathfrak{y}) = |\varkappa - \mathfrak{y}|,$$

$\forall\ \varkappa, \mathfrak{y} \in \mathcal{Y}$. *This metric space is complete.*
Define a multivalued operator $\Gamma : \mathcal{Y} \to \mathcal{C}(\mathcal{Y})$ by

$$\Gamma(\varkappa) = \begin{cases} \left\{1, \frac{1}{2^{n+1}}\right\}, & \text{when } \varkappa = \frac{1}{2^n}, \quad n \in \mathbb{N}, \\ \left\{0, \frac{1}{2}\right\}, & \text{when } \varkappa = 0. \end{cases}$$

This mapping fails to satisfy Nadler's contraction condition. For example,

$$\mathcal{H}\big(\Gamma(\tfrac{1}{2}), \Gamma(0)\big) = \frac{1}{2},$$

which is not bounded by the distance between the points:

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n},$$

which can be smaller than $\frac{1}{2}$ for all n, violating the classical contraction criterion.
Next, define the function $g : \mathcal{Y} \to \mathbb{R}$ as

$$f(\varkappa) = \rho(\varkappa, \Gamma(\varkappa)) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } \varkappa = \frac{1}{2^n}, \quad n \in \mathbb{N}, \\ 0, & \text{if } \varkappa = 0 \text{ or } \varkappa = 1. \end{cases}$$

It is straightforward to verify that g is continuous on $\mathcal{Y}$.
Moreover, for every $\varkappa \in \mathcal{Y}$, one can select $\mathfrak{y} \in \mathcal{I}^{\varkappa}_{0.7}$ such that

$$\rho(\mathfrak{y}, \Gamma(\mathfrak{y})) = \frac{1}{2} d(\varkappa, \mathfrak{y}),$$

confirming the generalized contraction condition in Theorem 1.2.2. Hence, the theorem ensures the existence of at least one fixed point for Γ in $\mathcal{Y}$.

Example 1.2.3. *Consider the set*

$$\mathcal{Y} = \left\{\frac{1}{2}, 1+\frac{1}{2}, \frac{1}{4}, 1+\frac{1}{4}, \ldots, \frac{1}{2^n}, 1+\frac{1}{2^n}, \ldots\right\} \cup \{0, 1\}$$

equipped with metric $\rho(\varkappa, \mathfrak{y}) = |\varkappa - \mathfrak{y}|\ \forall\ \varkappa, \mathfrak{y} \in \mathcal{Y}$. Clearly, $\mathcal{Y}, \rho$ is a metric space which is complete.

Consider a self-mapping $\Gamma : \mathcal{Y} \to \mathcal{Y}$ by

$$\Gamma(\varkappa) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } \varkappa = \frac{1}{2^n}, \\ 1 + \frac{1}{2^{n+1}}, & \text{if } \varkappa = 1 + \frac{1}{2^n},\ n = 1, 2, \ldots, \\ 0, & \text{if } \varkappa = 0, \\ 1, & \text{if } \varkappa = 1. \end{cases}$$

This mapping Γ is continuous, and the associated function $f(\varkappa) = |\varkappa - \Gamma(\varkappa)|$ is therefore continuous on $\mathcal{Y}$. Additionally, we observe that

$$\rho(\Gamma(\varkappa), \Gamma^2(\varkappa)) = \frac{1}{2}\rho(\varkappa, \Gamma(\varkappa))\ \forall\ \varkappa \in \mathcal{Y},$$

which suggests a contractive-type behavior for the iteration of Γ. Hence, Γ has fixed points in $\mathcal{Y}$, namely $\Gamma(0) = 0$ and $\Gamma(1) = 1$.

Chapter 2

Existence of Best Proximity Points for Multivalued Non-Self Mappings

In the previous chapter, we studied the existence of fixed points for multivalued self-mappings, focusing on conditions under which a multivalued mapping admits a point $\varkappa \in \mathcal{Y}$ such that $\varkappa \in \Gamma(\varkappa)$. These results were grounded in fixed point theory, which gives important tools for analyzing nonlinear problems in mathematics and its applications. In this chapter, we extend our study to a broader and more general framework by exploring the occurrence of best proximity points for multivalued non-self mappings.

2.1 Introduction and preliminaries

The approximation theorem has various applications in different fields. Some notable applications include: The field of quantum mechanics, particularly in the context of quantum state approximation. Control theory to design control systems that approximate desired behaviors. In game theory, especially in the study of cooperative game models. Unlike self-mappings, where the domain and codomain are the same, non-self mappings are defined between two distinct subsets of a metric space. That is, given a multivalued mapping $\Gamma : U \to 2^V$, where U and V are non-empty subsets of a metric space $(\mathcal{Y}, d)$ and $U \cap V = \varnothing$, a fixed point does not exist in the classical sense because $\Gamma(\varkappa) \subseteq U$. Instead, we seek a best proximity point a point $\varkappa \in U$ such that the distance between $\varkappa$ and its image $\Gamma(\varkappa) \subseteq V$ is minimized: $\rho(\varkappa, \Gamma(\varkappa)) = \inf \{d\rho(a, b) : a \in U, b \in V\}$. This concept of best proximity points is a natural generalization of fixed points and arises in

numerous practical situations where an exact solution (i.e., a fixed point) may not exist, but a best approximation is desirable or sufficient. Consider the following popular best approximation theorem due to Ky Fan [16].

Theorem 2.1.1 (Fan [16])**.** *Consider the normed linear space $\mathcal{Y}$ and let U be its subset which is compact, convex, and non-empty. Suppose $\Gamma : U \to \mathcal{Y}$ is a continuous mapping. Then, $\exists$ a point $\varkappa \in U$ such that the distance between $\varkappa$ and $\Gamma(\varkappa)$ is equal to the shortest distance from $\Gamma(\varkappa)$ to the set U. That is,*

$$\|\varkappa - \Gamma(\varkappa)\| = \inf \{\|\Gamma(\varkappa) - a\| : a \in U\}.$$

This kind of point $\varkappa$ is known as a best approximation point of Γ in U.

Definition 2.1.1. *Consider the metric space $(\mathcal{Y}, \rho)$ and let U and V be its non-empty subsets. Suppose $\Gamma : U \to V$ be a mapping that is not necessarily a self-map. A point $\varkappa \in U$ is known as best proximity point of Γ if it minimizes the distance between the point and its image under Γ, that is,*

$$\rho(\varkappa, \Gamma\varkappa) = \operatorname{dist}(U, V),$$

where the distance between the sets U and V is given by

$$\operatorname{dist}(U, V) = \inf\{\rho(\varkappa, \mathfrak{y}) : \varkappa \in U, \mathfrak{y} \in V\}.$$

The main focus of best proximity point theory is to determine under what requirements the optimization problem $\min_{\varkappa \in U} \rho(\varkappa, \Gamma\varkappa)$ has a solution. This topic plays an important role in various optimization frameworks, particularly when direct fixed point results are not applicable. In past few years, there has been a growing body of research addressing the existence, uniqueness, and convergence of such points for different types of mappings and structures (see, for example, [1, 2, 5, 6, 12, 13, 15, 28, 30–32]).

Let us introduce the following notations for convenience:

$$\begin{aligned}
U_0 &= \{\varkappa \in U : \rho(\varkappa, \mathfrak{y}) = \operatorname{dist}(U, V) \text{ for some } \mathfrak{y} \in V\},\\
V_0 &= \{\mathfrak{y} \in V : \rho(\varkappa, \mathfrak{y}) = \operatorname{dist}(U, V) \text{ for some } \varkappa \in U\},\\
\mathcal{D}(\varkappa, V) &= \inf\{\rho(\varkappa, \mathfrak{y}) : \mathfrak{y} \in V\}, \quad \text{for all } \varkappa \in \mathcal{Y},\\
\mathcal{H}(U, V) &= \max\left\{\sup_{\varkappa \in U} \mathcal{D}(\varkappa, V), \sup_{\mathfrak{y} \in V} \mathcal{D}(\mathfrak{y}, U)\right\}.
\end{aligned}$$

Now, consider the case where $\Gamma : U \to 2^V$ is a multivalued mapping (i.e., $\Gamma(\varkappa) \subseteq V$ for each $\varkappa \in U$). If $U \cap V = \varnothing$, a fixed point cannot exist, and thus we always have $\mathcal{D}(\varkappa, \Gamma\varkappa) > 0$ for all $\varkappa \in U$. In such cases, one aims to determine conditions under which the following minimization problem: $\min_{\varkappa \in U} \mathcal{D}(\varkappa, \Gamma\varkappa)$ has a solution. Since $\mathcal{D}(\varkappa, \Gamma\varkappa) \geq \operatorname{dist}(U, V)$ for all $\varkappa \in U$, an optimal solution is attained at points in U where this lower bound is achieved. A point $\varkappa^* \in U$ is known as a *best proximity point* of the multivalued non-self mapping Γ if

$$\mathcal{D}(\varkappa^*, \Gamma\varkappa^*) = \operatorname{dist}(U, V).$$

Note 2. *If* $\operatorname{dist}(U, V) = 0$*, then the notion of best proximity point boils down to that of a fixed point of* Γ*.*

Definition 2.1.2 (Raj [27])**.** *Consider the metric space* $(\mathcal{Y}, \rho)$ *and let* (U, V) *be a pair of non-empty subsets of* $\mathcal{Y}$*. The pair* (U, V) *possesses the P-property if*

$$\begin{cases} \rho(\varkappa_1, \mathfrak{y}_1) = \operatorname{dist}(U, V) \\ \rho(\varkappa_2, \mathfrak{y}_2) = \operatorname{dist}(U, V) \end{cases} \Rightarrow \rho(\varkappa_1, \varkappa_2) = \rho(\mathfrak{y}_1, \mathfrak{y}_2)$$

where $\varkappa_1, \varkappa_2 \in U_0$ *and* $\mathfrak{y}_1, \mathfrak{y}_2 \in V_0$*.*

Example 2.1.1. *Consider the Hilbert space* $\mathcal{Y}$ *and let* U *and* V *be two non-empty subsets that are both closed and convex. In this setting, the pair* (U, V) *satisfies the so-called P-property.*

Example 2.1.2. *Consider the metric space* $(\mathcal{Y}, \rho)$ *and let* U *and* V *be non-empty subsets of* $\mathcal{Y}$ *and suppose that the set* U_0 *is non-empty and the distance between* U *and* V *is zero. Under these conditions, the pair* (U, V) *meets the requirements for the P-property.*

Example 2.1.3 (Abkar and Gabeleh [1])**.** *Consider a Banach space* $\mathcal{Y}$ *which is uniformly convex and let* U *and* V *be its bounded, closed, and convex subsets that are non-empty. Then the pair* (U, V) *is known to exhibit the P-property.*

2.2 Multivalued contraction for non-self mappings

We begin this section with the following theorem.

Theorem 2.2.1. *Consider the metric space $(\mathcal{Y}, \rho)$ which is complete and let U and V be its closed, non-empty subsets, with $U_0 \neq \varnothing$. Additionally, suppose that the pair (U, V) fulfills the P-property condition. Let $\Gamma : U \to 2^V$ be a multivalued mapping such that:*

$$\mathcal{H}(\Gamma\varkappa, \Gamma\mathfrak{y}) \leq \alpha\rho(\varkappa, \mathfrak{y}). \quad \forall\ \varkappa, \mathfrak{y} \in U \text{ and some } \alpha \in (0, 1) \tag{*}$$

Further suppose that: Each $\Gamma(\varkappa) \subseteq V$ is non-empty, bounded and closed, For each $\varkappa_0 \in U_0$, we have $\Gamma(\varkappa_0) \subseteq V_0$. Then the mapping Γ possesses a best proximity point in U.

Proof. Let us begin by selecting an initial point $\varkappa_0 \in U_0$, and select $\mathfrak{y}_0 \in \Gamma(\varkappa_0) \subseteq V_0$. By the definition of V_0, $\exists$ a point $\varkappa_1 \in U_0$ such that

$$\rho(\varkappa_1, \mathfrak{y}_0) = \text{dist}(U, V).$$

By the definition of the Hausdorff metric, $\exists\ \mathfrak{y}_1 \in \Gamma(\varkappa_1)$ such that

$$\rho(\mathfrak{y}_0, \mathfrak{y}_1) \leq \mathcal{H}(\Gamma(\varkappa_0), \Gamma(\varkappa_1)) + \alpha.$$

Since $\Gamma(\varkappa_1) \subseteq V_0$, one can find $\varkappa_2 \in U_0$ satisfying

$$\rho(\varkappa_2, \mathfrak{y}_1) = \text{dist}(U, V),$$

and similarly, $\mathfrak{y}_2 \in \Gamma(\varkappa_2)$ such that

$$\rho(\mathfrak{y}_1, \mathfrak{y}_2) \leq \mathcal{H}(\Gamma(\varkappa_1), \Gamma(\varkappa_2)) + \alpha^2.$$

This construction can be continued inductively. For each $n \in \mathbb{N}$, $\exists$ a point $\varkappa_n \in U_0$ with

$$\rho(\varkappa_n, \mathfrak{y}_{n-1}) = \text{dist}(U, V),$$

and a point $\mathfrak{y}_n \in \Gamma(\varkappa_n)$ such that

$$\rho(\mathfrak{y}_{n-1}, \mathfrak{y}_n) \leq \mathcal{H}(\Gamma(\varkappa_{n-1}), \Gamma(\varkappa_n)) + \alpha^n.$$

Because $(U,\, V)$ fulfills the P-property and both

$$\rho(\varkappa_n, \mathfrak{y}_{n-1}) = \operatorname{dist}(U,\, V), \quad \rho(\varkappa_{n+1}, \mathfrak{y}_n) = \operatorname{dist}(U,\, V),$$

we have

$$\rho(\varkappa_n, \varkappa_{n+1}) = \rho(\mathfrak{y}_{n-1}, \mathfrak{y}_n), \quad \text{for all } n \in \mathbb{N}.$$

By contraction condition:

$$\begin{aligned}
\rho(\varkappa_n, \varkappa_{n+1}) &= \rho(\mathfrak{y}_{n-1}, \mathfrak{y}_n) \\
&\leq \mathcal{H}(\Gamma(\varkappa_{n-1}), \Gamma(\varkappa_n)) + \alpha^n \\
&\leq \alpha\rho(\varkappa_{n-1}, \varkappa_n) + \alpha^n \\
&= \alpha\rho(\mathfrak{y}_{n-2}, \mathfrak{y}_{n-1}) + \alpha^n \\
&\leq \alpha[\mathcal{H}(\Gamma(\varkappa_{n-2}), \Gamma(\varkappa_{n-1})) + \alpha^{n-1}] + \alpha^n \\
&\leq \alpha^2\rho(\varkappa_{n-2}, \varkappa_{n-1}) + 2\alpha^n \\
&\leq \cdots \leq \alpha^n\rho(\varkappa_0, \varkappa_1) + n\alpha^n.
\end{aligned}$$

Summing over n, we find:

$$\sum_{n=1}^{\infty} \rho(\varkappa_n, \varkappa_{n+1}) < \infty,$$

This shows that the sequence $\{\varkappa_n\}$ is Cauchy in the complete set U, and therefore it converges to a limit point $\varkappa^* \in U$. Moreover, since

$$\rho(\varkappa_n, \varkappa_{n+1}) = \rho(\mathfrak{y}_{n-1}, \mathfrak{y}_n),$$

the sequence $\{\mathfrak{y}_n\} \subseteq V$ is also Cauchy, which implies it converges to some $\mathfrak{y}^* \in V$.
Furthermore, because

$$\rho(\varkappa_{n+1}, \mathfrak{y}_n) = \operatorname{dist}(U,\, V) \quad \text{for all } n,$$

passing to the limit gives

$$\rho(\varkappa^*, \mathfrak{y}^*) = \operatorname{dist}(U,\, V).$$

We now assert that $\mathfrak{y}^* \in \Gamma(\varkappa^*)$. Since $\mathfrak{y}_n \in \Gamma(\varkappa_n)$, and $\varkappa_n \to \varkappa^*$, the continuity of Γ with

respect to the Hausdorff metric implies

$$\lim_{n\to\infty} \mathcal{D}(\mathfrak{y}_n, \Gamma(\varkappa^*)) \leq \lim_{n\to\infty} \mathcal{H}(\Gamma(\varkappa_n), \Gamma(\varkappa^*)) = 0.$$

Hence, $\mathfrak{y}^* \in \Gamma(\varkappa^*)$, and therefore

$$\mathcal{D}(\varkappa^*, \Gamma(\varkappa^*)) = \rho(\varkappa^*, \mathfrak{y}^*) = \operatorname{dist}(U, V).$$

Therefore, $\varkappa^*$ is a *best proximity point* of the multivalued mapping Γ. □

As a consequence of this result, one recovers the classical fixed point theorem by Nadler [21] by taking $U = V = \mathcal{Y}$.

Corollary 2.2.1. *Consider the metric space $(\mathcal{Y}, \rho)$ which is complete and let (U, V) be its non-empty closed subsets, $\operatorname{dist}(U, V) = 0$ and $U_0 \neq \varnothing$. Let $\Gamma_1, \Gamma_2 : U \to 2^V$ be two multivalued contraction-type mappings with $\alpha \in (0, 1)$, satisfying $\Gamma_i(U_0) \subseteq V_0$ for $i = 1, 2$. Assume also that for each $\varkappa \in U$, the set $\Gamma_i(\varkappa)$ is bounded and closed in V, and $\Gamma_i(\varkappa_0) \subseteq V_0$ for all $\varkappa_0 \in U_0$, $i = 1, 2$. If $F(\Gamma_1)$ and $F(\Gamma_2)$ represent the fixed point sets of Γ_1 and Γ_2, respectively, then the following holds:*

$$\mathcal{H}(F(\Gamma_1), F(\Gamma_2)) \leq \frac{1}{1-\alpha} \sup_{\varkappa\in U} \mathcal{H}(\Gamma_1(\varkappa), \Gamma_2(\varkappa)).$$

2.3 Feng-Liu type approach to the existence of best proximity points

Lemma 2.3.1. *Consider two metric spaces $(\mathcal{A}, \rho)$ and $(\mathcal{B}, \sigma)$, and suppose that the set-valued map $\Gamma : \mathcal{A} \to P(\mathcal{B})$ is upper semi-continuous. Define a mapping $g : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ by assigning to each pair $(\varkappa, \mathfrak{y})$ the value*

$$f(\varkappa, \mathfrak{y}) := \inf\{\sigma(\mathfrak{y}, \mathfrak{z}) : \mathfrak{z} \in \Gamma(\varkappa)\} = \sigma(\mathfrak{y}, \Gamma(\varkappa)).$$

Then, under these assumptions, the mapping g is lower semi-continuous on the product space $\mathcal{A} \times \mathcal{B}$.

Proof. Fix any $\alpha \in \mathbb{R}$, and define the set

$$O_\alpha = \{(\varkappa, \mathfrak{y}) \in \mathcal{A} \times \mathcal{B} : g(\varkappa, \mathfrak{y}) > \alpha\} = g^{-1}((\alpha, \infty)).$$

We aim to prove that O_α is open in $\mathcal{A} \times \mathcal{B}$. Choose any point $(\varkappa_0, \mathfrak{y}_0) \in O_\alpha$, which means $g(\varkappa_0, \mathfrak{y}_0) > \alpha$. Then, $\exists\ \varepsilon > 0$ such that

$$g(\varkappa_0, \mathfrak{y}_0) > \alpha + \varepsilon.$$

Now define the closed ball in $\mathcal{B}$ centered at $\mathfrak{y}_0$ with radius $\alpha + \varepsilon$:

$$\mathcal{B}_0 = \overline{B}_\sigma(\mathfrak{y}_0, \alpha + \varepsilon) = \{\mathfrak{y} \in \mathcal{B} : \sigma(\mathfrak{y}_0, \mathfrak{y}) \leq \alpha + \varepsilon\}.$$

Since $\mathcal{B}_0$ is closed and Γ is upper semi-continuous, the set $\Gamma^-(\mathcal{B}_0) = \{\varkappa \in \mathcal{A} : \Gamma(\varkappa) \cap \mathcal{B}_0 \neq \varnothing\}$ is closed in $\mathcal{A}$. Thus, the complement $\mathcal{A} \setminus \Gamma^-(\mathcal{B}_0)$ is open. From $g(\varkappa_0, \mathfrak{y}_0) > \alpha + \varepsilon$, we have $\Gamma(\varkappa_0) \cap \mathcal{B}_0 = \varnothing$, so $\varkappa_0 \in \mathcal{A} \setminus \Gamma^-(\mathcal{B}_0)$. Therefore, $\varkappa_0 \in A^* := \mathcal{A} \setminus \Gamma^-(\mathcal{B}_0)$, an open neighborhood of $\varkappa_0$. Also, define the open ball $B^* := B_\sigma(\mathfrak{y}_0, \varepsilon) \subset \mathcal{B}$, which is an open neighborhood of $\mathfrak{y}_0$. Consider the product neighborhood $A^* \times B^* \subset \mathcal{A} \times \mathcal{B}$.

We now show that $A^* \times B^* \subset O_\alpha$. Take any $(\varkappa, \mathfrak{y}) \in A^* \times B^*$. Then: - Since $\varkappa \in A^*$, we have $\Gamma(\varkappa) \cap \mathcal{B}_0 = \varnothing$, which means $\sigma(\mathfrak{y}_0, z) > \alpha + \varepsilon$ for all $z \in \Gamma(\varkappa)$, so $\sigma(\mathfrak{y}_0, \Gamma(\varkappa)) \geq \alpha + \varepsilon$. Since $\mathfrak{y} \in B^*$, $\sigma(\mathfrak{y}_0, \mathfrak{y}) < \varepsilon$.
Using the triangle inequality:

$$\sigma(\mathfrak{y}, \Gamma(\varkappa)) \geq \sigma(\mathfrak{y}_0, \Gamma(\varkappa)) - \sigma(\mathfrak{y}_0, \mathfrak{y}) > \alpha + \varepsilon - \varepsilon = \alpha.$$

Hence, $g(\varkappa, \mathfrak{y}) = \sigma(\mathfrak{y}, \Gamma(\varkappa)) > \alpha$, implying $(\varkappa, \mathfrak{y}) \in O_\alpha$. Therefore, O_α contains an open neighborhood around each of its points and is thus open in $\mathcal{A} \times \mathcal{B}$. As $\alpha \in \mathbb{R}$ was arbitrary, therefore g is lower semi-continuous. □

The following result is generalization of Theorem 2.2.1 using Feng-Liu type approach.

Theorem 2.3.1. *Let $U, V \subset \mathcal{Y}$ be two non-empty closed subsets of a metric space $(\mathcal{Y}, \rho)$ satisfying the P-property. Suppose $U_0 \neq \varnothing$, and let $\Gamma : U \to \mathcal{C}(V)$ be a mapping such that*

$\Gamma(U_0) \subset V_0$. *Assume that for all* $\varkappa \in U_0$ *and* $\mathfrak{y} \in \Gamma(\varkappa)$, $\exists\, \mathfrak{z} \in U_0$ *such that:*

$$\rho(\mathfrak{y}, \mathfrak{z}) = \rho(U, V), \tag{2.1}$$

$$\rho(\mathfrak{y}, \Gamma(\mathfrak{z})) \leq c\, \rho(\varkappa, \mathfrak{z}), \tag{2.2}$$

for some constant $c \in (0, 1)$. *Then, if the function* $f(\varkappa, \mathfrak{y}) = \rho(\mathfrak{y}, \Gamma(\varkappa))$ *is lower semi-continuous, the mapping* Γ *possesses a best proximity point in* U.

Proof. Let $\varkappa_0 \in U_0$ be arbitrary and choose $\mathfrak{y}_0 \in \Gamma(\varkappa_0)$. By assumption, there exists $\varkappa_1 \in U_0$ such that

$$\rho(\mathfrak{y}_0, \varkappa_1) = \rho(U, V), \quad \rho(\mathfrak{y}_0, \Gamma(\varkappa_1)) \leq c\, \rho(\varkappa_0, \varkappa_1).$$

Choose $b \in (c, 1)$, then $\exists\, \mathfrak{y}_1 \in \Gamma(\varkappa_1)$ such that

$$b\, \rho(\mathfrak{y}_0, \mathfrak{y}_1) \leq \rho(\mathfrak{y}_0, \Gamma(\varkappa_1)).$$

Again, from the assumptions, there exists $\varkappa_2 \in U_0$ such that

$$\rho(\mathfrak{y}_1, \varkappa_2) = \rho(U, V), \quad \rho(\mathfrak{y}_1, \Gamma(\varkappa_2)) \leq c\, \rho(\varkappa_1, \varkappa_2),$$

and we can find $\mathfrak{y}_2 \in \Gamma(\varkappa_2)$ satisfying

$$b\, \rho(\mathfrak{y}_1, \mathfrak{y}_2) \leq \rho(\mathfrak{y}_1, \Gamma(\varkappa_2)).$$

repeating in this manner, we get sequences $\{\varkappa_n\} \subset U_0$ and $\{\mathfrak{y}_n\} \subset V$ such that

$$\rho(\mathfrak{y}_n, \varkappa_{n+1}) = \rho(U, V), \tag{2.3}$$

$$\rho(\mathfrak{y}_n, \Gamma(\varkappa_{n+1})) \leq c\, \rho(\varkappa_n, \varkappa_{n+1}), \tag{2.4}$$

$$b\, \rho(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \leq \rho(\mathfrak{y}_n, \Gamma(\varkappa_{n+1})). \tag{2.5}$$

Since U and V satisfy the P-property, we also have

$$\rho(\varkappa_n, \varkappa_{n+1}) = \rho(\mathfrak{y}_{n-1}, \mathfrak{y}_n), \quad \text{for all } n \geq 1. \tag{2.6}$$

From (2.4)-(2.6), we deduce:

$$\begin{aligned}\rho(\varkappa_n, \varkappa_{n+1}) &= \rho(\mathfrak{y}_{n-1}, \mathfrak{y}_n) \\ &\leq \frac{1}{b}\rho(\mathfrak{y}_{n-1}, \Gamma(\varkappa_n)) \\ &\leq \frac{c}{b}\rho(\varkappa_{n-1}, \varkappa_n).\end{aligned}$$

Iteratively applying this inequality gives:

$$\rho(\varkappa_n, \varkappa_{n+1}) \leq \left(\frac{c}{b}\right)^n \rho(\varkappa_0, \varkappa_1),$$

which implies

$$\lim_{n\to\infty} \rho(\varkappa_n, \varkappa_{n+1}) = 0.$$

By using (2.4) and (2.6), we also obtain:

$$\lim_{n\to\infty} \rho(\mathfrak{y}_n, \Gamma(\varkappa_{n+1})) = 0 \tag{2.7}$$

and

$$\lim_{n\to\infty} \rho(\mathfrak{y}_n, \mathfrak{y}_{n+1}) = 0.$$

These estimates imply that $\{\varkappa_n\}$ and $\{\mathfrak{y}_n\}$ are Cauchy sequences. As U and V are closed subsets of the metric space $(\mathcal{Y}, \rho)$ which is complete, $\exists\ \varkappa^* \in U$ and $\mathfrak{y}^* \in V$ such that

$$\varkappa_n \to \varkappa^*, \quad \mathfrak{y}_n \to \mathfrak{y}^* \quad \text{as } n \to \infty.$$

Taking the limit in (2.3) gives:

$$\rho(\varkappa^*, \mathfrak{y}^*) = \rho(U, V). \tag{2.8}$$

Since $f(\varkappa, \mathfrak{y}) = \rho(\mathfrak{y}, \Gamma(\varkappa))$ is l.s.c and $(\varkappa_{n+1}, \mathfrak{y}_n) \to (\varkappa^*, \mathfrak{y}^*)$, we apply semicontinuity and the previous limit:

$$\begin{aligned}\rho(\mathfrak{y}^*, \Gamma(\varkappa^*)) &= f(\varkappa^*, \mathfrak{y}^*) \\ &\leq \liminf f(\varkappa_{n+1}, \mathfrak{y}_n) \\ &= \liminf \rho(\mathfrak{y}_n, \Gamma(\varkappa_{n+1}))\end{aligned}$$

$$= 0.$$

This implies $\mathfrak{y}^* \in \mathit{\Gamma}(\mathit{\varkappa}^*)$. From (2.8), we have

$$\rho(U, V) \leq \rho(\mathit{\varkappa}^*, \mathit{\Gamma}(\mathit{\varkappa}^*)) \leq \rho(\mathit{\varkappa}^*, \mathfrak{y}^*) = \rho(U, V),$$

hence,

$$\rho(\mathit{\varkappa}^*, \mathit{\Gamma}(\mathit{\varkappa}^*)) = \rho(U, V),$$

which means $\mathit{\varkappa}^*$ is a *best proximity point* of $\mathit{\Gamma}$ in U. □

Remark 2.3.1. *The conclusion of Theorem 2.3.1 can be applied to deduce Theorem 2.2.1, which corresponds to the principal result established by Abkar and Gabeleh [3].*

To illustrate this, consider an arbitrary point $\mathit{\varkappa} \in U_0$ and a point $\mathfrak{y} \in \mathit{\Gamma}(\mathit{\varkappa})$. Given that $\mathit{\Gamma}(\mathit{\varkappa}) \subset V_0$, $\exists$ some $\mathfrak{z} \in U_0$ such that:

$$\rho(\mathfrak{y}, \mathfrak{z}) = \rho(U, V).$$

Moreover, as $\mathfrak{y} \in \mathit{\Gamma}(\mathit{\varkappa})$, and assuming contraction inequality (*) holds, we obtain:

$$\rho(\mathfrak{y}, \mathit{\Gamma}(\mathfrak{z})) \leq H(\mathit{\Gamma}(\mathit{\varkappa}), \mathit{\Gamma}(\mathfrak{z})) \leq c\, \rho(\mathit{\varkappa}, \mathfrak{z}),$$

which confirms condition (2.2). Furthermore, as $\mathit{\Gamma}$ is continuous, it is automatically upper semi-continuous. From Lemma 2.3.1, we know that the function $f(\mathit{\varkappa}, \mathfrak{y}) = \rho(\mathfrak{y}, \mathit{\Gamma}(\mathit{\varkappa}))$ is lower semi-continuous. Hence, all conditions of Theorem 2.3.1 are fulfilled. Consequently, the mapping $\mathit{\Gamma}$ admits a best proximity point in U.

Example 2.3.1. *Consider the set $\mathcal{Y} = \mathbb{R}^2$ associated with the standard Euclidean metric, and define*

$$U = \{(1, a) : a \in \mathbb{R}\},$$
$$V = \{(0, b) : b \in \mathbb{R}\}.$$

It is evident that both U and V are closed subsets of $\mathcal{Y}$. The distance between them is

$$\rho(U, V) = 1,$$

and we also have $U_0 = U$ and $V_0 = V$. Moreover, the pair (U, V) satisfies the P-property. Define the mapping $\Gamma : U \to \mathcal{C}(V)$ as follows:

$$\Gamma(1, a) = \{(0, b) : b \geq a\}.$$

Clearly, $\Gamma(U_0) = V_0$.
Take an arbitrary point $\varkappa = (1, a) \in U_0$ and any $\mathfrak{y} = (0, b) \in \Gamma(\varkappa) = \Gamma(1, a)$. Consider the point $\mathfrak{z} = (1, b) \in U_0$. Then we get

$$\rho(\mathfrak{y}, \mathfrak{z}) = \rho((0, b), (1, b)) = 1 = \rho(U, V),$$
$$\rho(\mathfrak{y}, \Gamma(\mathfrak{z})) = \rho((0, b), \Gamma(1, b)) = 0 \leq c \cdot \rho(\varkappa, \mathfrak{z}) = c \cdot \rho((1, a), (1, b)),$$

for some constant $c \in (0, 1)$. Hence, all the requirements of Theorem 2.3.1 are fulfilled, which guarantees that Γ possesses a best proximity point in U.
***Note.** Since Γ does not map into $\mathcal{CB}(V)$ therefore theorem 2.2.1 cannot be applied in this context.*

Corollary 2.3.1. *Consider the metric space $(\mathcal{Y}, \rho)$ which is complete and a mapping $\Gamma : \mathcal{Y} \to \mathcal{C}(\mathcal{Y})$ such that for each $\varkappa \in \mathcal{Y}$ and every $\mathfrak{y} \in \Gamma(\varkappa)$, the inequality*

$$\rho(\mathfrak{y}, \Gamma(\mathfrak{y})) \leq c \cdot \rho(\varkappa, \mathfrak{y})$$

holds for some constant c with $0 < c < 1$. Assume that the function $g(\varkappa) := \rho(\varkappa, \Gamma(\varkappa))$ is lower semi-continuous on $\mathcal{Y}$. Under these conditions, the mapping Γ possesses a fixed point in $\mathcal{Y}$.

Chapter 3

Best Proximity Point Results for Cyclic Contractions

This chapter is dedicated to the study of some important results for best proximity points for cyclic contractions, which are mappings that alternate between two non-empty subsets U and V of a metric space $(\mathcal{Y}, d)$. Specifically, we consider a mapping $\Gamma : U \cup V \to U \cup V$ that satisfies a cyclic condition: $\Gamma(U) \subseteq V$ and $\Gamma(V) \subseteq U$, along with a generalized contraction condition of the form

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) \leq \lambda\, \rho(\varkappa, \mathfrak{y}) \quad \text{for all } \varkappa \in U,\ \mathfrak{y} \in V, \tag{3.1}$$

for some constant $\lambda \in (0, 1)$. When $U \cap V = \varnothing$, a fixed point in the classical sense cannot exist. Instead, we seek a point $\varkappa \in U \cup V$ such that

$$\rho(\varkappa, \Gamma\varkappa) = \operatorname{dist}(U, V),$$

which we define as a best proximity point. This framework has significant relevance in areas where approximating the closest feasible solution is critical, especially in constrained optimization and approximation theory.

In this chapter, we study existence results for best proximity points under cyclic contraction conditions. We adopt and generalize techniques inspired by classical fixed point theorems, adapting them to the context of non-self mappings. The results obtained here not only extend existing theorems but also provide a foundation for further exploration of best proximity point theory in more general settings.

3.1 Preliminaries

Consider the metric space $(\mathcal{Y}, \rho$ and let $U \subseteq \mathcal{Y}$. We define the metric projection of a point $\varkappa \in \mathcal{Y}$ onto the set U as:

$$P_U(\varkappa) = \{\mathfrak{y} \in \mathcal{Y} : \rho(\varkappa, \mathfrak{y}) = \rho(\varkappa, U)\}.$$

Lemma 3.1.1 (Kirk et al. [18]). *Consider the Banach space $\mathcal{Y}$ which is reflexive and let $U \subseteq \mathcal{Y}$ that is non-empty, bounded, convex and closed. Assume $V \subseteq \mathcal{Y}$ is also non-empty, convex and closed. Then the sets U_0 and V_0 are guaranteed to be non-empty. Moreover, the following inclusions hold:*

$$P_V(U_0) \subseteq V_0 \quad \textit{and} \quad P_U(V_0) \subseteq U_0.$$

Definition 3.1.1. *Consider the metric space $(\mathcal{Y}, \rho)$ and let $K \subset \mathcal{Y}$, then K is called* boundedly compact *if any sequence in K that remains bounded has at least one subsequence converging to a point that also lies in K.*

Theorem 3.1.1 (Kirk et al. [17]). *Consider the metric space $(\mathcal{Y}, \rho)$ which is complete and let U and V be its non-empty closed subsets. Consider a mapping $\Gamma : U \cup V \to U \cup V$ that fulfills the cyclic condition: namely, $\Gamma(U) \subseteq V$ and $\Gamma(V) \subseteq U$. Suppose $\exists$ a constant $\lambda \in (0, 1)$ such that*

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) \leq \lambda\, \rho(\varkappa, \mathfrak{y}) \quad \textit{for every } \varkappa \in U,\ \mathfrak{y} \in V.$$

Under these conditions, the sets U and V must have a non-empty intersection, and Γ has exactly one fixed point in $U \cap V$.

Theorem 3.1.2 (Eldered and Veeramani [14]). *Consider the metric space $(\mathcal{Y}, \rho)$ which is complete and let U and V be its non-empty closed subsets, and let $\Gamma : U \cup V \to U \cup V$ be a cyclic contraction. Suppose that an initial point $\varkappa_0 \in U$ is given, and define the sequence $\{\varkappa_n\}$ by $\varkappa_{n+1} = \Gamma\varkappa_n\ \forall\ n \in \mathbb{N}$. If the subsequence $\{\varkappa_{2n-1}\}$ has a subsequence convergent in U, then $\exists$ a point $\varkappa \in U$ such that*

$$\rho(\varkappa, \Gamma\varkappa) = \operatorname{dist}(U, V).$$

3.2 Best proximity point results for cyclic contractions

The section start with the following definitions.

Definition 3.2.1. *Consider the set $\mathcal{Y}$, and let U and V be its non-empty subsets, and let $\Gamma : U \cup V \to U \cup V$ be a function. If the image of U under Γ lies entirely in V, and the image of V lies entirely in U, that is,*

$$\Gamma(U) \subseteq V \quad \textit{and} \quad \Gamma(V) \subseteq U,$$

then such a function Γ is known as a cyclic function.

Definition 3.2.2. *Consider the metric space $\mathcal{Y}$ and let U and V be its non-empty subsets. A mapping $\Gamma : U \cup V \to U \cup V$ is known as a* cyclic contraction *if it fulfills the following two properties:*

1. *V contains the image of U under Γ , and U contains the image of V ; that is, $\Gamma(U) \subseteq V$ and $\Gamma(V) \subseteq U$.*
2. *$\exists$ a constant $\lambda \in (0, 1)$ such that for all $\varkappa \in U$ and $\mathfrak{y} \in V$,*

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) \leq \lambda\, \rho(\varkappa, \mathfrak{y}) + (1 - \lambda)\ \mathrm{dist}(U, V).$$

Note that condition (2) implies that Γ is nonexpansive in the sense that

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) \leq \rho(\varkappa, \mathfrak{y}), \quad \forall \varkappa \in U, \mathfrak{y} \in V.$$

Moreover, inequality (2) can also be represented as:

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) - \mathrm{dist}(U, V) \leq \lambda \cdot (\rho(\varkappa, \mathfrak{y}) - \mathrm{dist}(U, V)).$$

Proposition 3.2.1. *Consider the metric space $\mathcal{Y}$ and let U and V be its non-empty subsets. Suppose that $\Gamma : U \cup V \to U \cup V$ is a cyclic contraction. Then, for any point $\varkappa_0 \in U \cup V$, the sequence defined by $\varkappa_{n+1} = \Gamma\varkappa_n$ satisfies*

$$\lim_{n\to\infty} \rho(\varkappa_n, \Gamma\varkappa_n) = \mathrm{dist}(U, V).$$

Proof. We observe:

$$\begin{aligned}\rho(\varkappa_n, \varkappa_{n+1}) &\leq \lambda\rho(\varkappa_{n-1}, \varkappa_n) + (1-\lambda)\,\mathrm{dist}(U, V),\\ &\leq \lambda\left[\lambda\rho(\varkappa_{n-2}, \varkappa_{n-1}) + (1-\lambda)\,\mathrm{dist}(U, V)\right] + (1-\lambda)\,\mathrm{dist}(U, V),\\ &= \lambda^2\rho(\varkappa_{n-2}, \varkappa_{n-1}) + (1-\lambda^2)\,\mathrm{dist}(U, V).\end{aligned}$$

Proceeding by induction, we get:

$$\rho(\varkappa_n, \varkappa_{n+1}) \leq \lambda^n\rho(\varkappa_1, \varkappa_0) + (1-\lambda^n)\,\mathrm{dist}(U, V).$$

Hence,

$$\lim_{n\to\infty} \rho(\varkappa_n, \varkappa_{n+1}) = \mathrm{dist}(U, V).$$

□

Proposition 3.2.2. *Consider the metric space $\mathcal{Y}$ which is complete and let U and V be its non-empty closed subsets. Suppose $\Gamma : U \cup V \to U \cup V$ is a cyclic contraction, and let $\varkappa_0 \in U$ with $\varkappa_{n+1} = \Gamma\varkappa_n$. If the even-indexed subsequence $\{\varkappa_{2n}\} \subseteq U$ has a convergent subsequence, then $\exists\ \varkappa \in U$ such that*

$$\rho(\varkappa, \Gamma\varkappa) = \mathrm{dist}(U, V).$$

Proof. Let $\{\varkappa_{2n_k}\}$ be a subsequence of $\{\varkappa_{2n}\}$ converging to some $\varkappa \in U$. Then:

$$\begin{aligned}\mathrm{dist}(U, V) &\leq \rho(\varkappa, \varkappa_{2n_k-1})\\ &\leq \rho(\varkappa, \varkappa_{2n_k}) + \rho(\varkappa_{2n_k}, \varkappa_{2n_k-1}).\end{aligned}$$

As $\varkappa_{2n_k} \to \varkappa$ and $\rho(\varkappa_{2n_k}, \varkappa_{2n_k-1}) \to \mathrm{dist}(U, V)$, it follows that:

$$\rho(\varkappa, \varkappa_{2n_k-1}) \to \mathrm{dist}(U, V).$$

Also, by the properties of Γ,

$$\mathrm{dist}(U, V) \leq \rho(\varkappa_{2n_k}, \Gamma\varkappa) \leq \rho(\varkappa_{2n_k-1}, \varkappa),$$

and thus $\rho(\varkappa, \Gamma\varkappa) = \mathrm{dist}(U, V)$. □

Proposition 3.2.3. *Let U and V be two non-empty subsets of a metric space $\mathcal{Y}$, and suppose $\Gamma : U \cup V \to U \cup V$ is a cyclic contraction. Then, for any initial point $\varkappa_0 \in U \cup V$ and the sequence $\varkappa_{n+1} = \Gamma\varkappa_n$, both sequences $\{\varkappa_{2n}\}$ and $\{\varkappa_{2n+1}\}$ are bounded.*

Proof. Let us take a point $\varkappa_0 \in U$. From Proposition 3.2.1, we know that

$$\lim_{n\to\infty} \rho(\varkappa_{2n}, \varkappa_{2n+1}) = \operatorname{dist}(U, V),$$

so it suffices to show that $\{\varkappa_{2n+1}\}$ is bounded. Suppose, to the contrary, that $\{\varkappa_{2n+1}\}$ is unbounded. Then $\exists$ an integer N_0 such that

$$\rho(\Gamma^2\varkappa_0, \Gamma^{2N_0+1}\varkappa_0) > M \quad \text{and} \quad \rho(\Gamma^2\varkappa_0, \Gamma^{2N_0-1}\varkappa_0) \leq M,$$

for some constant M chosen such that

$$M > \max\left\{\frac{2\rho(\varkappa_0, \Gamma\varkappa_0)}{1/\lambda^2 - 1} + \operatorname{dist}(U, V),\ \rho(\Gamma^2\varkappa_0, \Gamma\varkappa_0)\right\}.$$

Using the cyclic contraction condition, we estimate:

$$\begin{aligned}\frac{M - \operatorname{dist}(U, V)}{\lambda^2} + \operatorname{dist}(U, V) &< \rho(\varkappa_0, \Gamma^{2N_0-1}\varkappa_0) \\ &\leq \rho(\varkappa_0, \Gamma^2\varkappa_0) + \rho(\Gamma^2\varkappa_0, \Gamma^{2N_0-1}\varkappa_0) \\ &\leq 2\rho(\varkappa_0, \Gamma\varkappa_0) + M,\end{aligned}$$

leading to a contradiction. Hence, $\{\varkappa_{2n+1}\}$ is bounded, and thus $\{\varkappa_{2n}\}$ is also bounded. □

Theorem 3.2.1. *Consider the metric space $\mathcal{Y}$ and let U and V be its non-empty closed subsets, and let $\Gamma : U \cup V \to U \cup V$ be a mapping that fulfills the cyclic contraction property. If one of the sets U and V is boundedly compact, then $\exists$ a point $\varkappa \in U \cup V$ for which*

$$\rho(\varkappa, \Gamma\varkappa) = \operatorname{dist}(U, V).$$

Corollary 3.2.1. *Assume that U and V are non-empty closed subsets within the normed linear space $\mathcal{Y}$, and let $\Gamma : U \cup V \to U \cup V$ be a mapping that satisfies the cyclic contraction property. If the linear span of either U or V is contained in a finite-dimensional subspace of $\mathcal{Y}$, then*

$\exists\, \varkappa \in U \cup V$ *such that*

$$\rho(\varkappa, \Gamma\varkappa) = \operatorname{dist}(U, V).$$

Remark 3.2.1. *Corollary 3.2.1 does not extend to the case where both U and V span infinite-dimensional subspaces of $\mathcal{Y}$.*

The supportive example for given results.

Example 3.2.1. *Let $\lambda \in (0,1)$ and define subsets U and V of ℓ^p, where $1 \leq p \leq \infty$, by*

$$U = \left\{(1+\lambda^{2n})e_{2n} : n \in \mathbb{N}\right\}, \quad V = \left\{(1+\lambda^{2m-1})e_{2m-1} : m \in \mathbb{N}\right\}.$$

Define a map Γ on $U \cup V$ as follows:

$$\Gamma\big((1+\lambda^{2n})e_{2n}\big) = (1+\lambda^{2n+1})e_{2n+1}, \quad \Gamma\big((1+\lambda^{2m-1})e_{2m-1}\big) = (1+\lambda^{2m})e_{2m}.$$

Then Γ is a cyclic contraction on $U \cup V$.

Proof. For $p = \infty$, the contraction condition is easy to verify. For $1 \leq p < \infty$, we compute

$$\operatorname{dist}(U, V) = 2^{1/p}.$$

Using the triangle inequality in ℓ^p:

$$\begin{aligned}
&\big((1+\lambda^{2n+1})^p + (1+\lambda^{2m})^p\big)^{1/p} \\
&= \big((\lambda + \lambda^{2n+1} + (1-\lambda))^p + (\lambda + \lambda^{2m} + (1-\lambda))^p\big)^{1/p} \\
&\leq \big((\lambda + \lambda^{2n+1})^p + (\lambda + \lambda^{2m})^p\big)^{1/p} + 2^{1/p}(1-\lambda) \\
&\leq \lambda\left((1+\lambda^{2n})^p + (1+\lambda^{2m-1})^p\right)^{1/p} + 2^{1/p}(1-\lambda).
\end{aligned}$$

Hence, Γ satisfies the cyclic contraction inequality. □

Remark 3.2.2. *Note that both U and V are closed, yet $U_0 = V_0 = \varnothing$, so no best proximity point exists in this case.*

Lemma 3.2.1. *Consider the Banach space $\mathcal{Y}$ which is uniformly convex and let U be its non-empty subset which is convex and closed and let V be closed subset of $\mathcal{Y}$. Suppose $\exists$ sequences $\{\varkappa_n\}$ and $\{\mathfrak{z}_n\}$ in U, and a sequence $\{\mathfrak{y}_n\}$ in V, such that:*

(i) $\|\mathfrak{z}_n - \mathfrak{y}_n\| \to \text{dist}(U, V)$,

(ii) For every $\varepsilon > 0$ $\exists$ N_0 *such that* $\|\varkappa_m - \mathfrak{y}_n\| \leq \text{dist}(U, V) + \varepsilon$ *for all* $m > n \geq N_0$.

Then for every $\varepsilon > 0$, $\exists$ N_1 *such that* $\|\varkappa_m - \mathfrak{z}_n\| < \varepsilon$ $\forall$ $m > n \geq N_1$.

Lemma 3.2.2. *Under the same assumptions as above Lemma 3.2.1, if*

(i) $\|\varkappa_n - \mathfrak{y}_n\| \to \text{dist}(U, V)$,

(ii) $\|\mathfrak{z}_n - \mathfrak{y}_n\| \to \text{dist}(U, V)$,

then $\|\varkappa_n - \mathfrak{z}_n\| \to 0$.

Corollary 3.2.2. *Consider the Banach space* $\mathcal{Y}$ *which is uniformly convex and let* U *be its subset which is non-empty, closed, and convex, and let* V *be its closed subset. Let* $\{\varkappa_n\}$ *be a sequence in* U *and* $\mathfrak{y}_0 \in V$ *such that* $\|\varkappa_n - \mathfrak{y}_0\| \to \text{dist}(U, V)$. *Then* $\varkappa_n \to P_U(\mathfrak{y}_0)$.

Theorem 3.2.2. *Consider the Banach space* $\mathcal{Y}$ *which is uniformly convex and let* U *and* V *be its closed convex subsets which are non-empty. If* $\Gamma : U \cup V \to U \cup V$ *is a cyclic contraction, then* $\exists$ *a unique point* $\varkappa \in U$ *such that*

$$\|\varkappa - \Gamma\varkappa\| = \text{dist}(U, V).$$

Furthermore, for $\varkappa_0 \in U$ *and* $\varkappa_{n+1} = \Gamma\varkappa_n$, *the sequence* $\{\varkappa_{2n}\}$ *converges to this point.*

Proof. If $\text{dist}(U, V) = 0$, then $U \cap V \neq \varnothing$ and the result follows from Banach's contraction principle. Assume $\text{dist}(U, V) > 0$.
From the properties of cyclic contractions, $\|\varkappa_{2n} - \Gamma\varkappa_{2n}\| \to \text{dist}(U, V)$ and $\|\Gamma^2\varkappa_{2n} - \Gamma\varkappa_{2n}\| \to \text{dist}(U, V)$. Then, Lemma 3.2.1 implies $\|\varkappa_{2n} - \varkappa_{2(n+1)}\| \to 0$ and similarly for $\Gamma\varkappa_{2n}$.
We claim:

$$\|\varkappa_{2m} - \Gamma\varkappa_{2n}\| \leq \text{dist}(U, V) + \varepsilon. \quad \forall\, \varepsilon > 0, \exists\, N_0, \forall\, m > n \geq N_0$$

Assuming the contrary leads to contradiction via the contraction property. Thus $\{\varkappa_{2n}\}$ is Cauchy and converges to some $\varkappa \in U$ with

$$\|\varkappa - \Gamma\varkappa\| = \text{dist}(U, V).$$

Uniqueness follows: if $\varkappa, \mathfrak{y} \in U$ satisfy the condition and $\varkappa \neq \mathfrak{y}$, then $\Gamma^2\varkappa = \varkappa$ and $\Gamma^2\mathfrak{y} = \mathfrak{y}$, leading to contradictions via triangle inequality and the strict convexity of norm. □

Remark 3.2.3. *If the convexity condition in Theorem 3.2.2 is removed, convergence and uniqueness may fail. For example, in $\mathbb{R}^4$, let $U = \{e_1, e_3\}$ and $V = \{e_2, e_4\}$. Define $\Gamma(e_i) = e_{i+1}$, with $e_{4+i} = e_i$. The map Γ is cyclic but does not yield a unique limit.*

3.3 Best proximity point results using (U, V)-Feng-Liu type cyclic contraction pair

Definition 3.3.1. *Consider the metric space $(\mathcal{Y}, \rho)$ and let U and V be its non-empty subsets. Let $\Gamma : U \to P(V)$ and $\mathcal{S} : V \to P(U)$ be set-valued mappings. Then the pair $(\Gamma, \mathcal{S})$ forms a Feng-Liu type cyclic contraction pair on (U, V) if $\exists$ constants $b, c \in (0, 1)$ with $c < b$ such that the following conditions are satisfied:*

(i) For each $\varkappa \in U$ with $\rho(\varkappa, \Gamma\varkappa) > \rho(U, V)$, $\exists\ \mathfrak{y} \in \Gamma_b^{\varkappa} := \{\mathfrak{y} \in \Gamma\varkappa : b\rho(\varkappa, \mathfrak{y}) \leq \rho(\varkappa, \Gamma\varkappa)\}$ such that

$$\rho(\mathfrak{y}, \mathcal{S}\mathfrak{y}) \leq c\,\rho(\varkappa, \mathfrak{y}) + (b - c)\rho(U, V).$$

(ii) For each $\mathfrak{y} \in V$ with $\rho(\mathfrak{y}, \mathcal{S}\mathfrak{y}) > \rho(U, V)$, $\exists\ \varkappa \in \mathcal{S}_b^{\mathfrak{y}} := \{\varkappa \in \mathcal{S}\mathfrak{y} : b\rho(\mathfrak{y}, \varkappa) \leq \rho(\mathfrak{y}, \mathcal{S}\mathfrak{y})\}$ such that

$$\rho(\varkappa, \Gamma\varkappa) \leq c\,\rho(\varkappa, \mathfrak{y}) + (b - c)\rho(U, V).$$

Definition 3.3.2. *Consider a metric space $(\mathcal{Y}, \rho)$ and let $U, V \subseteq \mathcal{Y}$ be non-empty subsets. Let $\Gamma : U \to P(V)$ and $\mathcal{S} : V \to P(U)$ be mappings, and fix $b \in (0, 1)$. Starting from an initial point $\varkappa_0 \in U$, a sequence $\{\varkappa_n\}$ in $\mathcal{Y}$ is known as $(\Gamma, \mathcal{S})$-*iterative sequence *if it satisfies:*

$$\varkappa_{2n+1} \in \Gamma_b^{\varkappa_{2n}} \quad \text{and} \quad \varkappa_{2n+2} \in \mathcal{S}_b^{\varkappa_{2n+1}}, \quad \forall n \in \mathbb{N}.$$

Theorem 3.3.1. *Consider the metric space $(\mathcal{Y}, \rho)$ and let $U, V \subseteq \mathcal{Y}$ be two non-empty subsets. Suppose $\Gamma : U \to P(V)$ and $\mathcal{S} : V \to P(U)$ are multivalued mappings that form a* Feng–Liu

type cyclic contraction *pair relative to* (U, V). *Then, under following conditions, either* Γ *or* $\mathcal{S}$ *admits at least one best proximity point.*

(a) The function $f(\varkappa) = \rho(\varkappa, \Gamma\varkappa)$ *is lower semi-continuous and every* $(\Gamma, \mathcal{S})$*-iterative sequence* $\{\varkappa_n\}$ *has a subsequence* $\{\varkappa_{2n}\}$ *converging in* U.

(b) The function $g(\mathfrak{y}) = \rho(\mathfrak{y}, \mathcal{S}\mathfrak{y})$ *is lower semi-continuous and every* $(\Gamma, \mathcal{S})$*-iterative sequence* $\{\varkappa_n\}$ *has a subsequence* $\{\varkappa_{2n+1}\}$ *converging in* V.

Proof. Let us begin with an arbitrary point $\varkappa_0 \in U$. If $\rho(\varkappa_0, \Gamma\varkappa_0) = \rho(U, V)$, then the result follows immediately. Otherwise, since $\rho(\varkappa_0, \Gamma\varkappa_0) > \rho(U, V)$, condition (i) ensures the existence of $\varkappa_1 \in \Gamma_b^{\varkappa_0}$ such that

$$\rho(\varkappa_1, \mathcal{S}\varkappa_1) \leq c\,\rho(\varkappa_0, \varkappa_1) + (b - c)\rho(U, V).$$

If $\rho(\varkappa_1, \mathcal{S}\varkappa_1) = \rho(U, V)$, we are done. If not, applying condition (ii), we find $\varkappa_2 \in \mathcal{S}_b^{\varkappa_1}$ satisfying

$$\rho(\varkappa_2, \Gamma\varkappa_2) \leq c\,\rho(\varkappa_1, \varkappa_2) + (b - c)\rho(U, V).$$

Continuing recursively, we generate a $(\Gamma, \mathcal{S})$-iterative sequence $\{\varkappa_n\}$ where:

$$\rho(\varkappa_{2n+2}, \Gamma\varkappa_{2n+2}) \leq c\,\rho(\varkappa_{2n+1}, \varkappa_{2n+2}) + (b - c)\rho(U, V), \tag{3.2}$$

$$\rho(\varkappa_{2n+1}, \mathcal{S}\varkappa_{2n+1}) \leq c\,\rho(\varkappa_{2n}, \varkappa_{2n+1}) + (b - c)\rho(U, V). \tag{3.3}$$

Since $\varkappa_{2n+1} \in \Gamma_b^{\varkappa_{2n}}$ and $\varkappa_{2n+2} \in \mathcal{S}_b^{\varkappa_{2n+1}}$, we know:

$$b\,\rho(\varkappa_{2n}, \varkappa_{2n+1}) \leq \rho(\varkappa_{2n}, \Gamma\varkappa_{2n}), \tag{3.4}$$

$$b\,\rho(\varkappa_{2n+1}, \varkappa_{2n+2}) \leq \rho(\varkappa_{2n+1}, \mathcal{S}\varkappa_{2n+1}). \tag{3.5}$$

From this, define the recurrence:

$$b\,\rho(\varkappa_n, \varkappa_{n+1}) \leq c\,\rho(\varkappa_{n-1}, \varkappa_n) + (b - c)\rho(U, V) \quad \text{for } n \geq 1.$$

Solving the recurrence yields:

$$\rho(\varkappa_n, \varkappa_{n+1}) \leq \left(\frac{c}{b}\right)^n \rho(\varkappa_0, \varkappa_1) + \left[1 - \left(\frac{c}{b}\right)^n\right] \rho(U, V).$$

As $n \to \infty$, this implies:

$$\lim_{n\to\infty} \rho(\varkappa_n, \varkappa_{n+1}) = \rho(U, V).$$

In particular:

$$\lim_{n\to\infty} \rho(\varkappa_{2n}, \varkappa_{2n+1}) = \rho(U, V), \quad \lim_{n\to\infty} \rho(\varkappa_{2n+1}, \varkappa_{2n+2}) = \rho(U, V).$$

Now suppose condition (a) holds. Then $\{\varkappa_{2n}\}$ possesses a convergent subsequence $\{\varkappa_{2n_\lambda}\}$ with $\varkappa_{2n_\lambda} \to \varkappa^* \in U$. By the lower semi-continuity of $f(\varkappa) = \rho(\varkappa, \Gamma\varkappa)$, we have:

$$\rho(U, V) \leq f(\varkappa^*) \leq \liminf_{\lambda\to\infty} f(\varkappa_{2n_\lambda}) \leq \liminf_{\lambda\to\infty} \rho(\varkappa_{2n_\lambda}, \varkappa_{2n_\lambda+1}) = \rho(U, V),$$

which implies $f(\varkappa^*) = \rho(U, V)$. Hence, $\varkappa^*$ is a best proximity point of Γ.
Similarly, if condition (b) holds, an analogous argument gives the existence of a point $\mathfrak{y}^* \in V$ such that $g(\mathfrak{y}^*) = \rho(\mathfrak{y}^*, \mathcal{S}\mathfrak{y}^*) = \rho(U, V)$, i.e., $\mathfrak{y}^*$ is a best proximity point of $\mathcal{S}$. □

Example 3.3.1. *Consider the standard metric space $\mathbb{R}$ with the usual distance metric ρ. Define subsets U and V of $\mathbb{R}$ by:*

$$U = \left\{\frac{1}{2^{2n}} : n \in \mathbb{N}\right\} \cup \{0\}, \quad V = \left\{\frac{1}{2^{2n+1}} : n \in \mathbb{N}\right\}.$$

Consider mappings $\Gamma : U \to P(V)$ and $\mathcal{S} : V \to P(U)$ by:

$$\Gamma\left(\frac{1}{2^{2n}}\right) = \left\{\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+3}}, \dots\right\}, \quad \Gamma(0) = V,$$

$$\mathcal{S}\left(\frac{1}{2^{2n+1}}\right) = \left\{\frac{1}{2^{2n+2}}, \frac{1}{2^{2n+4}}, \dots\right\}, \quad \textit{for all } n \in \mathbb{N}.$$

*Our claim is that $(\Gamma, \mathcal{S})$ forms a (U, V)-*Feng–Liu type cyclic contraction *pair with constants $c = \frac{1}{2}$ and $b \in \left(\frac{1}{2}, 1\right)$.*
Suppose $\varkappa \in U$ with $\rho(\varkappa, \Gamma\varkappa) > \rho(U, V) = 0$. Since $\varkappa \neq 0$, we have $\varkappa = \frac{1}{2^{2n}}$ for some $n \in \mathbb{N}$.

Choose $\mathfrak{y} = \frac{1}{2^{2n+1}} \in \Gamma\varkappa$. *Then:*

$$\begin{aligned} b\rho(\varkappa, \mathfrak{y}) &= b \cdot \rho\left(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\right) = b \cdot \frac{1}{2^{2n+1}} \\ &\leq \frac{1}{2^{2n+1}} = \rho\left(\frac{1}{2^{2n}}, \Gamma\varkappa\right) = \rho(\varkappa, \Gamma\varkappa), \\ \rho(\mathfrak{y}, \mathcal{S}\mathfrak{y}) &= \rho\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}}\right) = \frac{1}{2^{2n+2}} \\ &= \frac{1}{2}\rho\left(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\right) = \frac{1}{2}\rho(\varkappa, \mathfrak{y}) + (b-c)\rho(U, V). \end{aligned}$$

Similarly, for $\mathfrak{y} \in V$ *with* $\rho(\mathfrak{y}, \mathcal{S}\mathfrak{y}) > 0$, *there exists* $\varkappa \in \mathcal{S}_b\mathfrak{y}$ *satisfying:*

$$\rho(\varkappa, \Gamma\varkappa) \leq c\rho(\varkappa, \mathfrak{y}) + (b-c)\rho(U, V).$$

Hence, $(\Gamma, \mathcal{S})$ *is a* Feng–Liu type cyclic contraction pair.

Since U *is compact in* $\mathbb{R}$, *every sequence in* U *has a convergent subsequence, verifying condition (a) of Theorem 3.3.1. Define the function:*

$$f(\varkappa) = \rho(\varkappa, \Gamma\varkappa) = \begin{cases} \frac{1}{2^{2n+1}}, & \varkappa = \frac{1}{2^{2n}}, \\ 0, & \varkappa = 0 \end{cases}$$

is clearly lower semi-continuous.

Therefore, by Theorem 3.3.1, the mapping Γ *has a best proximity point in* U.

Remark 3.3.1. *Let* $(\mathcal{Y}, \rho)$ *be metric space, if the subsets* U *and* V *of* $(\mathcal{Y}, \rho)$ *are compact, then Theorem 3.3.1 leads directly to the following result.*

Corollary 3.3.1. *Consider the metric space* $(\mathcal{Y}, \rho)$ *and suppose* $U, V \subseteq \mathcal{Y}$ *are non-empty subsets which are compact. If* $\Gamma : U \to P(V)$ *and* $\mathcal{S} : V \to P(U)$ *form a* (U, V)-Feng-Liu type cyclic contraction pair, *then at least one of the mappings* Γ or $\mathcal{S}$ *admits a best proximity point under either of the following assumptions:*

(a) *The function* $f(\varkappa) = \rho(\varkappa, \Gamma\varkappa)$ *is lower semi-continuous on* U.

(b) *The function* $g(\mathfrak{y}) = \rho(\mathfrak{y}, \mathcal{S}\mathfrak{y})$ *is lower semi-continuous on* V.

Corollary 3.3.2. *By setting* $U = V = \mathcal{Y}$ *in Theorem 3.3.1 or the previous corollary, we get the following result. Suppose* $(\mathcal{Y}, \rho)$ *be a metric space, and let* $\Gamma, \mathcal{S} : \mathcal{Y} \to P(\mathcal{Y})$ *are two multivalued*

maps forming a $(\mathcal{Y}, \mathcal{Y})$-Feng-Liu type cyclic contraction pair. *Then, either* Γ or $\mathcal{S}$ *possesses a best proximity point provided one of the conditions below holds:*

(a) $f(\varkappa) = \rho(\varkappa, \Gamma\varkappa)$ *is lower semi-continuous and the sequence* $\{\varkappa_{2n}\}$ *possesses a convergent subsequence for every* $(\Gamma, \mathcal{S})$-iterative *sequence* $\{\varkappa_n\}$.

(b) $g(\mathfrak{y}) = \rho(\mathfrak{y}, \mathcal{S}\mathfrak{y})$ *is lower semi-continuous and the sequence* $\{\varkappa_{2n+1}\}$ *possesses a convergent subsequence for every* $(\Gamma, \mathcal{S})$*-iterative sequence* $\{\varkappa_n\}$.

Corollary 3.3.3. *Consider the metric space* $(\mathcal{Y}, \rho)$ *which is compact and let* $\Gamma, \mathcal{S} : \mathcal{Y} \to P(\mathcal{Y})$ *be multivalued mappings forming a* $(\mathcal{Y}, \mathcal{Y})$-Feng–Liu type cyclic contraction pair*, then:*

(i) *If* $f(\varkappa) = \rho(\varkappa, \Gamma\varkappa)$ *is lower semi-continuous, then* Γ *possesses a fixed point.*

(ii) *If* $g(\mathfrak{y}) = \rho(\mathfrak{y}, \mathcal{S}\mathfrak{y})$ *is lower semi-continuous, then* $\mathcal{S}$ *possesses a fixed point.*

Chapter 4

Finding Best Proximity Points for Multivalued F-Contractions

The primary objective of this chapter is to study a global optimization problem through the lens of best proximity point theory in the setting of a complete metric space. Our focus is on multivalued mappings, where the mapping assigns a set of values rather than a single value to each point in the domain. These mappings naturally arise in variational analysis, differential inclusions, game theory, and many other disciplines. In this chapter, we study the existence of best proximity points for multivalued mappings using Wardowski's technique, which generalizes classical contraction principles and provides a broader framework for analyzing non-self mappings. Specifically, we study a best proximity point theorem for multivalued mappings by employing the concept of F-contractions in conjunction with the weak P-property. Furthermore, we present an illustrative example in which the standard P-property fails to hold, yet the weak P-property is satisfied, thereby demonstrating the wider applicability of our result.

4.1 Preliminaries

Definition 4.1.1 ([11]). *Consider the metric space $(\mathcal{Y}, \rho)$. A multivalued map $\Gamma : \mathcal{Y} \to \mathcal{CB}(\mathcal{Y})$ is called as* $\mathcal{H}$-continuous *at a point $\varkappa_0 \in \mathcal{Y}$ if for every sequence $\{\varkappa_n\} \subset \mathcal{Y}$ converging to $\varkappa_0$, that is,*

$$\lim_{n\to\infty} \rho(\varkappa_n, \varkappa_0) = 0,$$

the images $\Gamma\varkappa_n$ approach $\Gamma\varkappa_0$ in the Hausdorff metric, meaning

$$\lim_{n\to\infty} \mathcal{H}(\Gamma\varkappa_n, \Gamma\varkappa_0) = 0.$$

Thus, convergence of points in $\mathcal{Y}$ guarantees convergence of their images under Γ with respect to the Hausdorff distance.

Note 3. *A multivalued contraction Γ is $\mathcal{H}$-continuous.*

Remark 4.1.1. *(i) If a multivalued mapping Γ is Hausdorff-continuous ($\mathcal{H}$-continuous) at each point in a subset $U \subseteq \mathcal{Y}$, then Γ is called as continuous on U.*

(ii) Every multivalued contraction mapping Γ is $\mathcal{H}$-continuous.

Definition 4.1.2 (Wardowski [29])**.** *Let $F : (0, +\infty) \to \mathbb{R}$ be a function satisfying the following properties:*

(F1) F is strictly increasing;

(F2) For any sequence $\{\varkappa_n\} \subset (0, +\infty)$,

$$\lim_{n\to\infty} \varkappa_n = 0 \iff \lim_{n\to\infty} F(\varkappa_n) = -\infty;$$

(F3) $\exists$ a constant $\lambda \in (0, 1)$ such that

$$\lim_{\varkappa\to 0^+} \varkappa^{\lambda} F(\varkappa) = 0.$$

*Let $\mathcal{F}$ be the collection of all mappings F fulfilling the conditions mentioned above. Consider the metric space $(\mathcal{Y}, \rho)$. A mapping $\Gamma : \mathcal{Y} \to \mathcal{Y}$ is known as F-*contraction *if $\exists$ a constant $\tau > 0$ and a function $F \in \mathcal{F}$ such that for every $\varkappa, \mathfrak{y} \in \mathcal{Y}$, whenever*

$$\rho(\Gamma\varkappa, \Gamma\mathfrak{y}) > 0,$$

the following inequality holds:

$$\tau + F(\rho(\Gamma\varkappa, \Gamma\mathfrak{y})) \leq F(\rho(\varkappa, \mathfrak{y})).$$

Definition 4.1.3 (Altun et al. [7]). *Consider the metric space $(\mathcal{Y}, \rho)$. A multivalued mapping $\Gamma : \mathcal{Y} \to \mathcal{CB}(\mathcal{Y})$ is known as multivalued F-contraction if $\exists$ a constant $\tau > 0$ and a function $F \in \mathcal{F}$ such that the following inequality holds:*

$$\tau + F(\mathcal{H}(\Gamma\varkappa, \Gamma\mathfrak{y})) \leq F(\rho(\varkappa, \mathfrak{y}))$$

for all $\varkappa, \mathfrak{y} \in \mathcal{Y}$ whenever $\Gamma\varkappa \neq \Gamma\mathfrak{y}$.

Remark 4.1.2. *A multivalued F-contraction is $\mathcal{H}$-continuous.*

Lemma 4.1.1 ([9, 10]). *Consider the metric space $(\mathcal{Y}, \rho)$ and let $U, V \in \mathcal{CB}(\mathcal{Y})$. Then the following hold:*

1. *For any $\varkappa \in \mathcal{Y}$ and $\mathfrak{y} \in V$, the distance satisfies $\mathcal{D}(\varkappa, V) \leq \rho(\varkappa, \mathfrak{y})$.*
2. *For every $\varkappa \in U$, we have $\mathcal{D}(\varkappa, V) \leq \mathcal{H}(U, V)$.*

Lemma 4.1.2 (Nadler [21]). *Let $U, V \in \mathcal{CB}(\mathcal{Y})$, and fix $\varkappa \in U$. Then $\forall\ \varepsilon > 0$, $\exists\ \mathfrak{y} \in V$ such that*

$$\rho(\varkappa, \mathfrak{y}) \leq \mathcal{H}(U, V) + \varepsilon.$$

However, there may not always exist a point $\mathfrak{y} \in V$ satisfying

$$\rho(\varkappa, \mathfrak{y}) \leq \mathcal{H}(U, V).$$

Nevertheless, if V is compact, then such a point $\mathfrak{y} \in V$ does exist for which

$$\rho(\varkappa, \mathfrak{y}) \leq \mathcal{H}(U, V).$$

4.2 Best proximity point result for multivalued F-contraction

The following theorem is the existence result of best proximity point for multivalued F-contractions associated with weak P-property.

Theorem 4.2.1. *Consider the complete metric space $(\mathcal{Y}, \rho)$ and let U and V be two non-empty closed subsets of $\mathcal{Y}$ such that $U_0 \neq \varnothing$, and the pair (U, V) satisfies the weak P-property. Assume that $\Gamma : U \to \mathcal{CB}(V)$ is a multivalued F-contraction for which $\Gamma(\varkappa)$ is compact for every*

$\varkappa \in U$, *and that* $\Gamma(\varkappa) \subseteq V_0$ *whenever* $\varkappa \in U_0$. *Under these conditions, the mapping* Γ *admits a best proximity point.*

Proof. Choose $\varkappa_0 \in U_0$ and select $\mathfrak{y}_0 \in \Gamma(\varkappa_0) \subset V_0$ such that

$$\rho(\varkappa_0, \mathfrak{y}_0) = \rho(U, V). \tag{4.1}$$

If $\mathfrak{y}_0 \in \Gamma(\varkappa_1)$ for some $\varkappa_1 \in U_0$, then $\rho(U, V) \leq \mathcal{D}(\varkappa_1, \Gamma(\varkappa_1)) \leq \rho(\varkappa_1, \mathfrak{y}_0) = \rho(U, V)$, hence $\varkappa_1$ is a best proximity point of Γ. Otherwise, assume $\mathfrak{y}_0 \notin \Gamma(\varkappa_1)$. As $\Gamma(\varkappa_1)$ is compact, by Lemma 4.1.2, $\exists\, \mathfrak{y}_1 \in \Gamma(\varkappa_1)$ such that

$$0 < \rho(\mathfrak{y}_0, \mathfrak{y}_1) \leq \mathcal{H}(\Gamma(\varkappa_0), \Gamma(\varkappa_1)).$$

As F is strictly increasing,

$$F(\rho(\mathfrak{y}_0, \mathfrak{y}_1)) \leq F(\mathcal{H}(\Gamma(\varkappa_0), \Gamma(\varkappa_1))) \leq F(\rho(\varkappa_0, \varkappa_1)) - \tau. \tag{4.2}$$

Now, choose $\varkappa_2 \in U_0$ such that

$$\rho(\varkappa_2, \mathfrak{y}_1) = \rho(U, V). \tag{4.3}$$

Using the weak P-property and equations (4.1) and (4.3),

$$\rho(\varkappa_1, \varkappa_2) \leq \rho(\mathfrak{y}_0, \mathfrak{y}_1). \tag{4.4}$$

From (4.2) and (4.4), we obtain

$$F(\rho(\varkappa_1, \varkappa_2)) \leq F(\rho(\mathfrak{y}_0, \mathfrak{y}_1)) \leq F(\rho(\varkappa_0, \varkappa_1)) - \tau.$$

Continue this process. If $\mathfrak{y}_1 \notin \Gamma(\varkappa_2)$, then from Lemma 4.1.2 $\exists\, \mathfrak{y}_2 \in \Gamma(\varkappa_2)$ such that

$$0 < \rho(\mathfrak{y}_1, \mathfrak{y}_2) \leq \mathcal{H}(\Gamma(\varkappa_1), \Gamma(\varkappa_2)).$$

Then

$$F(\rho(\varkappa_2, \varkappa_3)) \leq F(\rho(\mathfrak{y}_1, \mathfrak{y}_2)) \leq F(\rho(\varkappa_1, \varkappa_2)) - \tau \leq F(\rho(\varkappa_0, \varkappa_1)) - 2\tau.$$

By proceeding this way, we get sequences $\{\varkappa_n\} \subset U_0$ and $\{\mathfrak{y}_n\} \subset V_0$ such that:

- $\mathfrak{y}_n \in \Gamma(\varkappa_n)$ and $\rho(\varkappa_{n+1}, \mathfrak{y}_n) = \rho(U, V)$,
- $F(\rho(\varkappa_n, \varkappa_{n+1})) \leq F(\rho(\varkappa_0, \varkappa_1)) - n\tau$.

Define $a_n = \rho(\varkappa_n, \varkappa_{n+1})$. Taking the limit as $n \to \infty$ gives:

$$\lim_{n\to\infty} F(a_n) = -\infty \Rightarrow \lim_{n\to\infty} a_n = 0.$$

From (F3), $\exists\ \lambda \in (0, 1)$ such that $a_n^\lambda F(a_n) \to 0$. Then

$$a_n^\lambda F(a_n) - a_n^\lambda F(a_0) \leq -n a_n^\lambda \tau \Rightarrow \lim_{n\to\infty} n a_n^\lambda = 0.$$

Hence $\exists n_0 \in \mathbb{N}$ such that $a_n \leq \frac{1}{n^{1/\lambda}}$ for $n \geq n_0$. For $m > n \geq n_0$,

$$\rho(\varkappa_m, \varkappa_n) \leq \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/\lambda}}.$$

Since the series converges for $\lambda \in (0, 1)$, the sequence $\{\varkappa_n\}$ is Cauchy in U_0. As U_0 is closed in complete $\mathcal{Y}$, there exists $\mathfrak{q} \in U_0$ such that $\varkappa_n \to \mathfrak{q}$.

Since Γ is $\mathcal{H}$-continuous, we obtain

$$\lim_{n\to\infty} \mathcal{H}(\Gamma(\varkappa_n), \Gamma(\mathfrak{q})) = 0.$$

Similarly, $\mathfrak{y}_n \to \mathfrak{y} \in V_0$ and $\rho(\varkappa_{n+1}, \mathfrak{y}_n) = \rho(U, V)$ implies $\rho(\mathfrak{q}, \mathfrak{y}) = \rho(U, V)$.

As $\mathfrak{y}_n \in \Gamma(\varkappa_n)$, and $\mathcal{H}(\Gamma(\varkappa_n), \Gamma(\mathfrak{q})) \to 0$, we have $\rho(\mathfrak{y}, \Gamma(\mathfrak{q})) = 0$, and since $\Gamma(\mathfrak{q})$ is closed, $\mathfrak{y} \in \Gamma(\mathfrak{q})$. Thus,

$$\rho(U, V) \leq \mathcal{D}(\mathfrak{q}, \Gamma(\mathfrak{q})) \leq \rho(\mathfrak{q}, \mathfrak{y}) = \rho(U, V) \Rightarrow \mathcal{D}(\mathfrak{q}, \Gamma(\mathfrak{q})) = \rho(U, V).$$

Hence, $\mathfrak{q}$ is a best proximity point of Γ. □

Example 4.2.1. *Let $\mathcal{Y} = \mathbb{R}$ be the space of real numbers equipped with the standard metric $\rho(\varkappa, \mathfrak{y}) = |\varkappa - \mathfrak{y}|$. Consider the subsets $U = [5, 6]$ and $V = [-6, -5]$. The distance between the sets is $\rho(U, V) = 10$, and the minimal points are $U_0 = \{5\}$, $V_0 = \{-5\}$.*
Define a set-valued mapping $\Gamma : U \to \mathcal{CB}(V)$ by

$$\Gamma(\varkappa) = \left[\frac{-\varkappa - 5}{2}, -5\right] \quad \text{for all } \varkappa \in [5, 6].$$

It follows that $\Gamma(5) = \{-5\}$, showing $\Gamma(\varkappa) \subseteq V_0$ for all $\varkappa \in U_0$.
We show that Γ fulfills the condition of a multivalued F-contraction. Assume $\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y})) > 0$. Then:

$$\begin{aligned}
\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y})) &= \mathcal{H}\left(\left[\frac{-\varkappa - 5}{2}, -5\right], \left[\frac{-\mathfrak{y} - 5}{2}, -5\right]\right) \\
&= \left|\frac{-\varkappa - 5}{2} - \frac{-\mathfrak{y} - 5}{2}\right| \\
&= \frac{|\varkappa - \mathfrak{y}|}{2} \\
&= \frac{\rho(\varkappa, \mathfrak{y})}{2} \\
&< \rho(\varkappa, \mathfrak{y}).
\end{aligned}$$

Using the logarithmic function $F(t) = \ln t$ defined on $(0, +\infty)$, we get

$$\ln(\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y}))) < \ln(\rho(\varkappa, \mathfrak{y})),$$

which implies that for any $\tau \in (0, \ln 2]$,

$$\tau + F(\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y}))) \leq F(\rho(\varkappa, \mathfrak{y})).$$

Hence, Γ is an F-contraction.
Additionally, the pair (U, V) meets the criteria for the weak P-property. Therefore, all assumptions of Theorem 4.2.1 are fulfilled, and it follows that $\varkappa = 5$ serves as a best proximity point of Γ.

Example 4.2.2. *Let $\mathcal{Y} = \mathbb{R}^2$ be equipped with the standard Euclidean metric ρ. Define the*

subsets

$$U = \{(-5,0),(0,1),(5,0)\}, \quad V = \left\{(\varkappa,\mathfrak{y}) \in \mathbb{R}^2 : \mathfrak{y} = 2 + \sqrt{2-\varkappa^2},\ \varkappa \in [-\sqrt{2},\sqrt{2}]\right\}.$$

It can be verified that the minimal distance between the two sets is

$$\rho(U, V) = \sqrt{3},$$

and the associated minimal sets are

$$U_0 = \{(0,1)\}, \quad V_0 = \left\{(\sqrt{2},2),(-\sqrt{2},2)\right\}.$$

Define a set-valued mapping $\Gamma : U \to \mathcal{CB}(V)$ *by:*

$$\Gamma(-5,0) = \{(-\sqrt{2},2),(-1,3)\}, \quad \Gamma(0,1) = \{(\sqrt{2},2)\}, \quad \Gamma(5,0) = \{(\sqrt{2},2),(1,3)\}.$$

It is straightforward to confirm that Γ *fulfills the criteria of a multivalued* F*-contraction with* $\tau = \ln 2$ *and* $F(t) = \ln t$ *for all* $t > 0$.

We observe:

$$\rho((0,1),(\sqrt{2},2)) = \rho((0,1),(-\sqrt{2},2)) = \sqrt{3} = \rho(U, V),$$

while

$$\rho((0,1),(0,1)) = 0 < \rho((\sqrt{2},2),(-\sqrt{2},2)) = 2\sqrt{2}.$$

Hence, although the pair (U, V) *does not fulfill the strict* P*-property, it does satisfy the weaker form, known as the weak* P*-property.*

As all requirements of Theorem 4.2.1 are fulfilled, we conclude that $(0,1)$ *is a best proximity point of the map* Γ*, as*

$$\rho((0,1),\Gamma(0,1)) = \sqrt{3} = \rho(U, V).$$

Chapter 5

Best Proximity Point Results for Multivalued Mappings on Partial Metric Spaces

In this chapter, we form new best proximity point results within the framework of 0-complete partial metric spaces by introducing a novel class of mixed multivalued contraction mappings. These findings serve as a broad generalization and enhancement of several well-known theorems available in existing literature. To validate the effectiveness of our main results, a significant illustrative example is provided, offering both insight and comparison. Moreover, we explore applications of the developed theorems to homotopy theory in a manner distinct from previously known approaches. As a consequence, we derive best proximity point theorems applicable to homotopic mappings.

5.1 Preliminaries

In [33], Reich asked the following question: Let $(\mathcal{Y}, \rho)$ be a metric space and $\Gamma : \mathcal{Y} \to \mathcal{CB}(\mathcal{Y})$ be a multivalued map satisfying $\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y})) \leq \mu(\rho(\varkappa, \mathfrak{y})) \cdot \rho(\varkappa, \mathfrak{y}) \; \forall \; \varkappa, \mathfrak{y} \in \mathcal{Y}$ with $\varkappa \neq \mathfrak{y}$, where '$\mu$' is *MT-function* defined below. Then does Γ has a fixed point? The answer to this question is affirmative, which is proved by Reich in [25].

Definition 5.1.1. *A function* $\mu : (0, \infty) \to [0, 1)$ *is termed an* MT-function *if it fulfills the condi-*

tion:

$$\limsup_{s \to r^+} \mu(s) < 1 \quad \textit{for all } r \geq 0.$$

The family of all such functions is denoted by ϕ*. Every monotone (increasing or decreasing) function is an* MT-function*, indicating the significance of this class.*

Theorem 5.1.1 (Reich [25])**.** *Consider the metric space* $(\mathcal{Y}, \rho)$ *which is complete and let* $\Gamma : \mathcal{Y} \to \mathcal{K}(\mathcal{Y})$ *be a multivalued mapping. Suppose*

$$\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y})) \leq \mu(\rho(\varkappa, \mathfrak{y})) \cdot \rho(\varkappa, \mathfrak{y})$$

$\forall\ \varkappa, \mathfrak{y} \in \mathcal{Y}$ *with* $\varkappa \neq \mathfrak{y}$*, where* $\mu : (0, \infty) \to [0, 1)$ *is a function fulfilling*

$$\limsup_{s \to r^+} \mu(s) < 1 \quad \textit{for every } r > 0.$$

Then Γ *possesses a fixed point.*

Theorem 5.1.2 (Mizoguchi and Takahashi [20])**.** *Consider the metric space* $(\mathcal{Y}, \rho)$ *which is complete and suppose* $\Gamma : \mathcal{Y} \to \mathcal{CB}(\mathcal{Y})$ *be a multivalued map. Assume that*

$$\mathcal{H}(\Gamma(\varkappa), \Gamma(\mathfrak{y})) \leq \mu(\rho(\varkappa, \mathfrak{y})) \cdot \rho(\varkappa, \mathfrak{y})$$

$\forall\ \varkappa, \mathfrak{y} \in \mathcal{Y}$ *with* $\varkappa \neq \mathfrak{y}$*, where* $\mu : (0, \infty) \to [0, 1)$ *satisfies*

$$\limsup_{s \to r^+} \mu(s) < 1 \quad \textit{for each } r \geq 0.$$

Then the mapping Γ *possesses a fixed point in* $\mathcal{Y}$*.*

Definition 5.1.2 (Matthews [19])**.** *Let* $\mathcal{W} \neq \varnothing$ *be any set and* $\sigma : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ *be a function. if the following hypotheses are fulfilled, then* σ *is a partial metric :*

(p1) $\sigma(\varkappa, \varkappa) = \sigma(\varkappa, \mathfrak{y}) = \sigma(\mathfrak{y}, \mathfrak{y})$ *if and only if* $\varkappa = \mathfrak{y}$*,*

(p2) $\sigma(\varkappa, \varkappa) \leq \sigma(\varkappa, \mathfrak{y})$*,*

(p3) $\sigma(\varkappa, \mathfrak{y}) = \sigma(\mathfrak{y}, \varkappa)$*,*

(p4) $\sigma(\varkappa, \mathfrak{z}) \leq \sigma(\varkappa, \mathfrak{y}) + \sigma(\mathfrak{y}, \mathfrak{z}) - \sigma(\mathfrak{y}, \mathfrak{y})$*,*

for all $\varkappa, \mathfrak{y}, \mathfrak{z} \in \mathcal{W}$. The pair $(\mathcal{W}, \sigma)$ is then called a partial metric space.

Remark 5.1.1. *Every metric is a partial metric, but the converse is not necessarily true.*

Definition 5.1.3 (Romaguera [26])**.** *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $\{\varkappa_n\}$ be any sequence in $\mathcal{W}$.*

(i) The sequence $\{\varkappa_n\}$ is known as 0-Cauchy sequence if $\lim_{n,m\to\infty} \sigma(\varkappa_n, \varkappa_m) = 0$.

(ii) The space $(\mathcal{W}, \sigma)$ is known as 0-complete if every sequence which is 0-Cauchy, converges to a point $\varkappa \in \mathcal{W}$ such that $\lim_{n\to\infty} \sigma(\varkappa_n, \varkappa) = \sigma(\varkappa, \varkappa) = 0$.

Remark 5.1.2. *Every 0-Cauchy sequence is also Cauchy in the usual sense, and thus every partial metric space which is complete is also 0-complete. However, the converse does not necessarily hold.*

Example 5.1.1. *Consider the set $\mathcal{W} = \mathbb{Q} \cap [0, \infty)$ equipped with the metric $\sigma : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ defined by*

$$\sigma(\varkappa, \mathfrak{y}) = \max\{\varkappa, \mathfrak{y}\} \quad \text{for all } \varkappa, \mathfrak{y} \in \mathcal{W}.$$

Then $(\mathcal{W}, \sigma)$ forms a partial metric space. However, this space is not complete with respect to the partial metric σ. Nevertheless, it is 0-complete, since every 0-Cauchy sequence in $\mathcal{W}$ converges to some point in $\mathcal{W}$ under the topology induced by σ.

Lemma 5.1.1 (Matthews [19])**.** *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $\{\varkappa_n\}$ be a sequence in $\mathcal{W}$.*

(i) $\{\varkappa_n\}$ is a Cauchy sequence in $(\mathcal{W}, \sigma)$ $\iff$ it is Cauchy in the induced metric space $(\mathcal{W}, \rho_\sigma)$.

(ii) $(\mathcal{W}, \sigma)$ is complete $\iff$ $(\mathcal{W}, \rho_\sigma)$ is complete.

(iii) For $\varkappa \in \mathcal{W}$,

$$\lim_{n\to\infty} \rho_\sigma(\varkappa_n, \varkappa) = 0 \iff \sigma(\varkappa, \varkappa) = \lim_{n\to\infty} \sigma(\varkappa_n, \varkappa) = \lim_{n,m\to\infty} \sigma(\varkappa_n, \varkappa_m).$$

It also follows that

$$\lim_{n\to\infty} \rho_\sigma(\varkappa_n, \varkappa) = 0, \quad \lim_{n\to\infty} \rho_\sigma(\mathfrak{y}_n, \mathfrak{y}) = 0 \Rightarrow \lim_{n\to\infty} \sigma(\varkappa_n, \mathfrak{y}_n) = \sigma(\varkappa, \mathfrak{y}).$$

Lemma 5.1.2 (Altun and Simsek [8]). *Let $(\mathcal{W}, \sigma)$ be a partial metric space and let $\varnothing \neq U \subseteq \mathcal{W}$. Then,*

$$\varkappa \in \overline{U} \iff \sigma(\varkappa, U) = \sigma(\varkappa, \varkappa),$$

where $\overline{U}$ is the closure of U in the topology induced by σ.

Lemma 5.1.3 (Abbas et al. [5]). *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $U, V \in \mathcal{CB}_\sigma(\mathcal{W})$. Then:*

(i) $\mathcal{H}_\sigma(U, U) \leq \mathcal{H}_\sigma(U, V)$,

(ii) $\mathcal{H}_\sigma(U, V) = \mathcal{H}_\sigma(V, U)$,

(iii) $\mathcal{H}_\sigma(U, V) \leq \mathcal{H}_\sigma(U, Z) + \mathcal{H}_\sigma(Z, V) - \inf_{\mathfrak{z} \in Z} \sigma(\mathfrak{z}, \mathfrak{z})$,

(iv) $\mathcal{H}_\sigma(U, V) = 0$ *implies* $U = V$.

Theorem 5.1.3 (Abbas et al. [5]). *Consider the partial metric space $(\mathcal{W}, \sigma)$ which is complete and let $\Psi : \mathcal{W} \to \mathcal{CB}_\sigma(\mathcal{W})$ be a multivalued map. If $\exists\ q \in [0, 1)$ such that*

$$\mathcal{H}_\sigma(\Psi\varkappa, \Psi\mathfrak{y}) \leq q\sigma(\varkappa, \mathfrak{y}) \quad \forall\ \varkappa, \mathfrak{y} \in \mathcal{W},$$

then Ψ possesses a fixed point.

Example 5.1.2. *Let $\mathcal{W} = [0, \infty)$ and define $\sigma(\varkappa, \mathfrak{y}) = \max\{\varkappa, \mathfrak{y}\}\ \forall\ \varkappa, \mathfrak{y} \in \mathcal{W}$. Then σ is a partial metric on $\mathcal{W}$. Let $U = [\varkappa, \infty)$ for some $\varkappa \in \mathcal{W}$, which is a closed subset in the topology induced by σ. Note that every such closed subset is unbounded, hence $\mathcal{CB}_\sigma(\mathcal{W}) = \varnothing$.*

Definition 5.1.4. *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $\varnothing \neq U, V \subseteq \mathcal{W}$ and $\Psi : U \to \mathcal{CB}_\sigma(V)$ be a multivalued mapping. Then a point $\varkappa^* \in U$ is a best proximity point of Ψ if*

$$\sigma(\varkappa^*, \Psi\varkappa^*) = \sigma(U, V).$$

Definition 5.1.5 (Gabeleh [4]). *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $\varnothing \neq U, V \subseteq \mathcal{W}$, and $U_0 \neq \varnothing$. Then, the pair (U, V) fulfills the weak P-property if for any $\varkappa_1, \varkappa_2 \in U_0$ and $\mathfrak{y}_1, \mathfrak{y}_2 \in V$ with*

$$\begin{cases} \sigma(\varkappa_1, \mathfrak{y}_1) = \sigma(U, V) \\ \sigma(\varkappa_2, \mathfrak{y}_2) = \sigma(U, V) \end{cases} \Rightarrow \sigma(\varkappa_1, \varkappa_2) \leq \sigma(\mathfrak{y}_1, \mathfrak{y}_2).$$

Where,

$$U_0 = \{\varkappa \in U : \sigma(\varkappa, \mathfrak{y}) = \sigma(U, V) \text{ for some } \mathfrak{y} \in V\}$$

and

$$V_0 = \{\mathfrak{y} \in V : \sigma(\varkappa, \mathfrak{y}) = \sigma(U, V) \text{ for some } \varkappa \in U\}.$$

Definition 5.1.6. *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $\varnothing \neq U, V \subset \mathcal{W}$ with $U_0 \neq \varnothing$. The pair (U, V) fulfills the weak P^*-property if for any $\varkappa_1 \neq \varkappa_2 \in U_0$ and $\mathfrak{y}_1, \mathfrak{y}_2 \in V_0$, the following conditions are satisfied:*

$$\sigma(\varkappa_1, \mathfrak{y}_1) = \sigma(U, V) \quad \text{and} \quad \sigma(\varkappa_2, \mathfrak{y}_2) = \sigma(U, V) \quad \Rightarrow \quad \sigma(\varkappa_1, \varkappa_2) \leq \sigma(\mathfrak{y}_1, \mathfrak{y}_2).$$

Remark 5.1.3. *If a pair (U, V) possesses the weak P-property, then it necessarily satisfies the weak P^*-property. However, the converse implication is not generally valid. To illustrate this, consider the space $(\mathcal{W}, \sigma)$ as defined in Example 5.1.2. If we take $U = \{2\}$ and $V = \{1\}$, it can be verified that (U, V) meets the criteria for the weak P^*-property but fails to satisfy the weak P-property.*

5.2 Existence of Best proximity points in 0-complete partial metric space

Lemma 5.2.1 (Abbas et al. [5])**.** *Consider the partial metric space $(\mathcal{W}, \sigma)$ and suppose $U, V \in \mathcal{CB}_\sigma(\mathcal{W})$ with $U \neq \varnothing \neq V$. Then, for any $a \in U$ and for each $\varepsilon > 0$, $\exists\ b \in V$ such that*

$$\sigma(a, b) \leq \mathcal{H}_\sigma(U, V) + \varepsilon.$$

Definition 5.2.1. *Consider the partial metric space $(\mathcal{W}, \sigma)$ and let $\varnothing \neq U, V \subseteq \mathcal{W}$. Consider a multivalued operator $\Psi : U \to V \cup \mathcal{CB}_\sigma(V)$. Ψ is known as* mixed multivalued contraction *if $\exists$ a function $\mu \in \mathcal{M}$ such that*

$$\mathcal{H}_\sigma(\Psi\, \varkappa, \Psi\, \mathfrak{y}) \leq \mu(\sigma(\varkappa, \mathfrak{y})) \cdot \sigma(\varkappa, \mathfrak{y}), \quad \forall\ \varkappa, \mathfrak{y} \in U. \tag{5.1}$$

Theorem 5.2.1. *Consider 0-complete the partial metric space $(\mathcal{W}, \sigma)$. Assume that $U, V \subseteq \mathcal{W}$*

are non-empty closed subsets with $U_0 \neq \varnothing$. *Assume that* $\Psi : U \to V \cup \mathcal{CB}_\sigma(V)$ *is a mixed multivalued contraction mapping satisfying* $\Psi(U_0) \subseteq V_0$, *and that the pair* (U, V) *possesses the weak* P^**-property. Then, the mapping* Ψ *admits a best proximity point* $\varkappa^* \in U$.

Proof. Consider $\beta : [0, \infty) \to [0, 1)$ defined by

$$\beta(t) = \begin{cases} \frac{1+\mu(t)}{2}, & t > 0, \\ 0, & t = 0. \end{cases}$$

Clearly, $\limsup_{r \to t^+} \beta(r) < 1$ for all $t \geq 0$.

Let $\varkappa_0 \in U_0$. We examine two cases:

Case I: $\Psi_{\varkappa_0}$ is a singleton. Then $\mathfrak{y}_0 = \Psi_{\varkappa_0} \in V$ and since $\mathfrak{y}_0 \in V_0$, there exists $\varkappa_1 \in U_0$ with

$$\sigma(\varkappa_1, \mathfrak{y}_0) = \sigma(U, V).$$

If $\varkappa_1 = \varkappa_0$, then $\varkappa_0$ is a best proximity point. If not, we claim that there is $\varkappa_2 \in U_0$ such that

$$\sigma(\varkappa_1, \varkappa_2) \leq \beta(\sigma(\varkappa_0, \varkappa_1)) \cdot \sigma(\varkappa_0, \varkappa_1).$$

Subcase I(a): $\Psi_{\varkappa_1}$ is a singleton. Let $\mathfrak{y}_1 = \Psi_{\varkappa_1} \in V_0$, so we find $\varkappa_2 \in U_0$ with

$$\sigma(\varkappa_2, \mathfrak{y}_1) = \sigma(U, V).$$

Using the weak P^*-property and the contractive inequality:

$$\begin{aligned} \sigma(\varkappa_1, \varkappa_2) &\leq \sigma(\mathfrak{y}_0, \mathfrak{y}_1) \\ &\leq \mathcal{H}_\sigma(\Psi_{\varkappa_0}, \Psi_{\varkappa_1}) \\ &\leq \mu(\sigma(\varkappa_0, \varkappa_1)) \cdot \sigma(\varkappa_0, \varkappa_1) \\ &\leq \beta(\sigma(\varkappa_0, \varkappa_1)) \cdot \sigma(\varkappa_0, \varkappa_1). \end{aligned}$$

Subcase I(b): $\Psi_{\varkappa_1}$ is not a singleton. Then by Lemma 5.2.1, there exists $\mathfrak{y}_1 \in \Psi_{\varkappa_1}$ such that

$$\sigma(\mathfrak{y}_0, \mathfrak{y}_1) \leq \mathcal{H}_\sigma(\Psi_{\varkappa_0}, \Psi_{\varkappa_1}) + \frac{1 - \mu(\sigma(\varkappa_0, \varkappa_1))}{2} \cdot \sigma(\varkappa_0, \varkappa_1).$$

Then,

$$\sigma(\mathfrak{y}_0, \mathfrak{y}_1) \leq \beta(\sigma(\varkappa_0, \varkappa_1)) \cdot \sigma(\varkappa_0, \varkappa_1).$$

Since $\mathfrak{y}_1 \in V_0$, there is $\varkappa_2 \in U_0$ with $\sigma(\varkappa_2, \mathfrak{y}_1) = \sigma(U, V)$. The inequality for $\sigma(\varkappa_1, \varkappa_2)$ follows as before.

Case II: $\Psi_{\varkappa_0}$ is not a singleton. Then by Lemma 5.2.1, we can choose $\mathfrak{y}_0 \in \Psi_{\varkappa_0}$ such that there exists $\varkappa_1 \in U_0$ with $\sigma(\varkappa_1, \mathfrak{y}_0) = \sigma(U, V)$. Proceed as in Case I.

By continuing this process, we define sequences $\{\varkappa_n\} \subseteq U_0$ and $\{\mathfrak{y}_n\} \subseteq V_0$ satisfying:

$$\begin{aligned}
\sigma(\varkappa_{n+1}, \mathfrak{y}_n) &= \sigma(U, V), \\
\sigma(\varkappa_n, \varkappa_{n+1}) &\leq \sigma(\mathfrak{y}_{n-1}, \mathfrak{y}_n), \\
\sigma(\mathfrak{y}_{n-1}, \mathfrak{y}_n) &\leq \beta(\sigma(\varkappa_{n-1}, \varkappa_n)) \cdot \sigma(\varkappa_{n-1}, \varkappa_n).
\end{aligned}$$

Thus, $\{\sigma(\varkappa_n, \varkappa_{n+1})\}$ is non-increasing and converges to $\gamma \geq 0$. The properties of β imply $\gamma = 0$, hence $\{\varkappa_n\}$ and $\{\mathfrak{y}_n\}$ are 0-Cauchy.

Because $\mathcal{W}$ is 0-complete and both U, V are closed, $\exists\ \varkappa^* \in U$ and $\mathfrak{y}^* \in V$ such that

$$\begin{aligned}
\sigma(\varkappa^*, \varkappa^*) &= \lim_{n\to\infty} \sigma(\varkappa_n, \varkappa^*) = 0, \\
\sigma(\mathfrak{y}^*, \mathfrak{y}^*) &= \lim_{n\to\infty} \sigma(\mathfrak{y}_n, \mathfrak{y}^*) = 0.
\end{aligned}$$

From Lemma 5.1.1(iii), we conclude

$$\sigma(\varkappa^*, \mathfrak{y}^*) = \sigma(U, V).$$

Finally, using continuity and contractiveness of Ψ, we conclude

$$\sigma(\varkappa^*, \Psi_{\varkappa^*}) = \sigma(U, V),$$

proving that $\varkappa^*$ is a best proximity point. □

Example 5.2.1. *Let $\mathcal{W} = \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n, \ldots\}$ be a countable collection. Consider a sequence $\{\varkappa_n\} \subset \mathbb{R}^+$ which is bounded above and satisfies:*

$$\lim_{n\to\infty} \varkappa_n =: \varkappa_\infty > \varkappa_1 > \varkappa_2 > 0, \quad \textit{and} \quad \varkappa_{n+1} < \varkappa_n \textit{ for all } n \geq 3.$$

An example of such a sequence is given by $\varkappa_1 = \frac{1}{5}, \varkappa_2 = \frac{1}{6}, \varkappa_n = 1 + \frac{1}{n}$ *for all* $n \geq 3$. *Define the mapping* $\sigma : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ *by:*

$$\sigma(\mathfrak{w}_i, \mathfrak{w}_j) = \begin{cases} \frac{\varkappa_i}{2} & \text{if } i = j, \\ \max\{\varkappa_i \varkappa_j\} & \text{if } i \neq j. \end{cases}$$

Then, the pair $(\mathcal{W}, \sigma)$ *forms a partial metric space. Moreover, as there are no* 0*-Cauchy sequences in* $(\mathcal{W}, \sigma)$*, it is* 0*-complete.*

Define two subsets of $\mathcal{W}$ *by* $U = \{\mathfrak{w}_{2k-1} \mid k \in \mathbb{N}\}$ *and* $V = \{\mathfrak{w}_{2k} \mid k \in \mathbb{N}\}$. *Note that for each* $\mathfrak{w}_n \in \mathcal{W}$, *the ball*

$$B_\sigma\left(\mathfrak{w}_n, \frac{\varkappa_n}{2}\right) = \{\mathfrak{w}_n\},$$

which shows that the topology τ_σ *induced by* σ *is discrete. Consequently, both* U *and* V *are closed.*

Observe that:

$$U_0 = \{\mathfrak{w}_1\}, \quad V_0 = \{\mathfrak{w}_2\}, \quad \text{and} \quad \sigma(U, V) = \varkappa_1.$$

Thus, the pair (U, V) *fulfills the weak* P^**-property. Define the multivalued mapping* $\Psi : U \to 2^V$ *as follows:*

$$\Psi(\mathfrak{w}_{2k-1}) = \begin{cases} \{\mathfrak{w}_2\}, & \text{if } k = 1, \\ (\mathcal{W}_{2k} \cap V), & \text{if } k \geq 2, \end{cases}$$

where $\mathcal{W}_n := \{\mathfrak{w}_n, \mathfrak{w}_{n+1}, \mathfrak{w}_{n+2}, \ldots\}$ *for all* $n \in \mathbb{N}$. *Then, clearly,* $\Psi(U_0) \subseteq V_0$.

Now, define a control function $\mu : (0, \infty) \to [0, 1)$ *by:*

$$\mu(r) = \begin{cases} \frac{\varkappa_{k+1}}{\varkappa_k}, & \text{if } r = \frac{\varkappa_k}{2} \text{ or } r = \varkappa_k \text{ for some } k \in \{1, 3, 5, \ldots\}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, as $\limsup_{s \to r^+} \mu(s) = 0 < 1$, μ *is a Mizoguchi-Takahashi type function. We now verify that* Ψ *is a* mixed multivalued contraction.

Let us consider three possible scenarios for $\mathfrak{w}_{2k-1}, \mathfrak{w}_{2l-1} \in U$*:*

Case 1. *If* $k = l$, *then*

$$\mathcal{H}(\Psi(\mathfrak{w}_{2k-1}), \Psi(\mathfrak{w}_{2l-1})) = \mathcal{H}(\mathcal{W}_{2k} \cap V, \mathcal{W}_{2k} \cap V) = \frac{\varkappa_{2k}}{2} = \mu(\sigma(\mathfrak{w}_{2k-1}, \mathfrak{w}_{2k-1})) \cdot \sigma(\mathfrak{w}_{2k-1}, \mathfrak{w}_{2k-1}).$$

Case 2. *If* $k = 1$ *and* $l > 1$*, then*

$$\mathcal{H}(\Psi(\mathfrak{w}_1), \Psi(\mathfrak{w}_{2l-1})) = \mathcal{H}(\{\mathfrak{w}_2\}, \mathcal{W}_{2l} \cap V) = \varkappa_{2l} = \mu(\sigma(\mathfrak{w}_1, \mathfrak{w}_{2l-1})) \cdot \sigma(\mathfrak{w}_1, \mathfrak{w}_{2l-1}).$$

Case 3. *If* $k \neq l$*,* $l > k \geq 2$*, then*

$$\mathcal{H}(\Psi(\mathfrak{w}_{2k-1}), \Psi(\mathfrak{w}_{2l-1})) = \mathcal{H}(\mathcal{W}_{2k} \cap V, \mathcal{W}_{2l} \cap V) = \varkappa_{2k} = \mu(\sigma(\mathfrak{w}_{2k-1}, \mathfrak{w}_{2l-1})) \cdot \sigma(\mathfrak{w}_{2k-1}, \mathfrak{w}_{2l-1}).$$

Thus, all hypothesis of Theorem 5.2.1 are fulfilled, and hence Ψ *has a best proximity point in* U*, which is* $\mathfrak{w}_1$*.*
Furthermore, observe that there exists no constant $c \in [0, 1)$ *such that*

$$\mathcal{H}(\Psi(\mathfrak{u}), \Psi(\mathfrak{v})) \leq c \cdot \sigma(\mathfrak{u}, \mathfrak{v}) \quad \textit{for all } \mathfrak{u}, \mathfrak{v} \in U.$$

To see this, assume such c *exists. Then, for* $\mathfrak{u} = \mathfrak{w}_{2k-1}, \mathfrak{v} = \mathfrak{w}_{2l-1}$ *with* $l > k > 0$*, we have:*

$$\mathcal{H}(\Psi(\mathfrak{w}_{2k-1}), \Psi(\mathfrak{w}_{2l-1})) = \varkappa_{2k} \leq c \cdot \varkappa_{2k-1} \Rightarrow \frac{\varkappa_{2k}}{\varkappa_{2k-1}} \leq c.$$

Taking the limit as $k \to \infty$ *yields* $1 \leq c$*, a contradiction. Hence, the contraction principle in [5] is not applicable here.*
Moreover, since $\sigma(\varkappa, \varkappa) = 0$ *in any metric space, there does not exist a control function* $\mu \in \phi$ *satisfying the assumptions in [3] for this example. Therefore, those fixed point results are not valid in this setting.*

Corollary 5.2.1. *Consider the partial metric space* $(\mathcal{W}, \sigma)$ *which is 0-complete and suppose* $U, V \subseteq \mathcal{W}$ *are its subsets which are non-empty and closed such that* $U_0 \neq \varnothing$*. Let* $\Psi : U \to V \cup \mathcal{CB}_\sigma(V)$ *be a* mixed multivalued mapping *such that* $\Psi(U_0) \subseteq V_0$*, and suppose that the pair* (U, V) *possesses the weak* P^**-property. If* $\exists$ *a constant* $q \in [0, 1)$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa), \Psi(\mathfrak{y})) \leq q \cdot \sigma(\varkappa, \mathfrak{y}) \quad \textit{for all } \varkappa, \mathfrak{y} \in U,$$

then Ψ *admits a best proximity point* $\varkappa^* \in U$*.*

Corollary 5.2.2. *Consider the partial metric space* $(\mathcal{W}, \sigma)$ *which is 0-complete and a mixed*

multivalued mapping $\Psi : \mathcal{W} \to \mathcal{W} \cup \mathcal{CB}_\sigma(\mathcal{W})$. *Suppose* $\exists$ *a function* $\mu \in \Phi$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa), \Psi(\mathfrak{y})) \leq \mu(\sigma(\varkappa, \mathfrak{y})) \cdot \sigma(\varkappa, \mathfrak{y}) \quad \forall\, \varkappa, \mathfrak{y} \in \mathcal{W}.$$

Then Ψ *possesses a fixed point* $\varkappa^* \in \mathcal{W}$.

Proof. Choose $U = V = \mathcal{W}$ in Theorem 5.2.1. Then $\exists\ \varkappa^* \in \mathcal{W}$ such that

$$\sigma(\varkappa^*, \Psi(\varkappa^*)) = \sigma(\mathcal{W}, \mathcal{W}).$$

Since $\sigma(\mathcal{W}, \mathcal{W}) \leq \sigma(\varkappa^*, \varkappa^*)$ and $\sigma(\mathcal{W}, \mathcal{W}) \leq \sigma(\Psi(\varkappa^*), \Psi(\varkappa^*))$, therefore

$$\sigma(\varkappa^*, \varkappa^*) = \sigma(\varkappa^*, \Psi(\varkappa^*)) = \sigma(\Psi(\varkappa^*), \Psi(\varkappa^*)),$$

implying that $\varkappa^* \in \Psi(\varkappa^*)$, i.e., $\varkappa^*$ is a fixed point. □

Corollary 5.2.3. *Consider the partial metric space* $(\mathcal{W}, \sigma)$ *which is 0-complete and let* $\Psi : \mathcal{W} \to \mathcal{W} \cup \mathcal{CB}_\sigma(\mathcal{W})$ *be a* mixed multivalued *mapping. If* $\exists$ *a constant* $q \in [0, 1)$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa), \Psi(\mathfrak{y})) \leq q \cdot \sigma(\varkappa, \mathfrak{y}) \quad \forall\, \varkappa, \mathfrak{y} \in \mathcal{W},$$

then Ψ *admits a fixed point* $\varkappa^* \in \mathcal{W}$.

5.3 Application in homotopy theory

Definition 5.3.1. *Let* (U, τ_1) *and* (V, τ_2) *be two topological spaces. Suppose* $f, g : U \to V$ *are mappings which are continuous. Then* f *and* g *are known as* homotopic *if* $\exists$ *a continuous function* $\Psi : U \times [0, 1] \to V$ *such that*

$$\Psi(\varkappa, 0) = f(\varkappa) \quad \textit{and} \quad \Psi(\varkappa, 1) = g(\varkappa) \quad \forall\, \varkappa \in U.$$

The function Ψ *is referred to as a* homotopy *between* f *and* g.

Example 5.3.1. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be functions that are continuous, and consider a function*

$\Psi : \mathbb{R} \times [0,1] \to \mathbb{R}$ *by*

$$\Psi(\varkappa, \mathfrak{t}) = (1-\mathfrak{t})f(\varkappa) + \mathfrak{t}g(\varkappa), \quad \forall\, \varkappa \in \mathbb{R},\, \mathfrak{t} \in [0,1].$$

Then Ψ *is a continuous mapping satisfying* $\Psi(\varkappa, 0) = f(\varkappa)$ *and* $\Psi(\varkappa, 1) = g(\varkappa)$. *Thus,* Ψ *represents a homotopy between* f *and* g.

Theorem 5.3.1. *Consider the partial metric space* $(\mathcal{W}, \sigma)$ *which is 0-complete and suppose* $U, V \subseteq \mathcal{W}$ *are its non-empty and closed subsets with* $U_0 \neq \varnothing$ *and* $U^* \subseteq U$ *is non-empty. Let the pair* (U, V) *fulfill the weak* P*-property. Consider a mixed multivalued mapping* $\Psi : U \times [0,1] \to V \cup \mathcal{CB}_\sigma(V)$ *such that:*

(i) For every $\varkappa \in U \setminus U^*$ *and* $\mathfrak{y} \in [0,1]$, *we have* $\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) > \sigma(U, V)$.

(ii) $\exists\, \mu \in \mathcal{H}$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\mathfrak{y}, z)) \leq \mu(\sigma(\varkappa, \mathfrak{y})) \cdot \sigma(\varkappa, \mathfrak{y})$$

$\forall\, \varkappa, \mathfrak{y} \in U$ *with* $\sigma(\varkappa, \mathfrak{y}) > 0$, *and* $\mathfrak{y}, z \in [0,1]$.

(iii) For each $\mathfrak{y}, \mathfrak{z} \in [0,1]$ *and* $\varkappa \in U$ *with* $\sigma(\varkappa, \varkappa) = 0$, $\exists$ *a function* $\kappa : [0,1] \to \mathbb{R}$ *which is continuous such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\varkappa, \mathfrak{z})) \leq q|\kappa(\mathfrak{y}) - \kappa(\mathfrak{z})|.$$

(iv) For any $\mathfrak{y} \in [0,1]$ *such that* $\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) = \sigma(U, V)$ *for some* $\varkappa \in A^*$, $\exists\, \varepsilon_\mathfrak{y} > 0$ *such that*

$$\Psi(U_0, \mathfrak{y}^*) \subseteq V_0 \quad \forall\, \mathfrak{y}^* \in (\mathfrak{y} - \varepsilon_\mathfrak{y}, \mathfrak{y} + \varepsilon_\mathfrak{y}).$$

(v) If $\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) = \sigma(U, V)$, *then* $\Psi(\varkappa, \mathfrak{y})$ *is a singleton.*

If the mapping $\Psi(\cdot, 0)$ *admits a* best proximity point *in* U, *then so does* $\Psi(\cdot, 1)$.

Example 5.3.2. *Consider the set* $\mathcal{Y} = [1, \infty)$ *and a function* $\sigma : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ *defined by*

$$\sigma(\varkappa, \mathfrak{y}) = \begin{cases} |\varkappa - \mathfrak{y}| & \text{if } \varkappa, \mathfrak{y} \in [1, 4], \\ \dfrac{\varkappa}{2} & \text{if } \varkappa = \mathfrak{y} \in \mathcal{Y} \setminus [1, 4], \\ \varkappa + \mathfrak{y} & \text{otherwise.} \end{cases}$$

Then, $(\mathcal{Y}, \sigma)$ *is a partial metric space which is* 0*-complete. Consider the subsets* $U = \{1\} \cup [5, 6]$ *and* $V = [3, 4]$*. Clearly, both* U *and* V *are closed, and the minimal distance between them is given by* $\sigma(U, V) = 4$*. Furthermore, the sets of minimal distance elements are* $U_0 = \{1\}$ *and* $V_0 = \{3\}$*. One can verify that the pair* (U, V) *fulfills the weak* P*-property.*
Define two mappings $f, g : U \to V$ *by*

$$f(\varkappa) = 3, \quad g(\varkappa) = 4 \quad \text{for all } \varkappa \in U.$$

Since both mappings are continuous, a homotopy between them is given by

$$\Psi(\varkappa, \mathfrak{y}) = (1 - \mathfrak{y}) f(\varkappa) + \mathfrak{y} g(\varkappa) = 3 + \mathfrak{y},$$

for $\mathfrak{y} \in [0, 1]$*. Let* $U^* = U_0 = \{1\}$*. It is immediate that hypotheses (i) and (v) of Theorem 5.3.1 are satisfied.*
Define auxiliary functions $\mu : (0, \infty) \to [0, 1)$ *and* $\kappa : [0, 1] \to \mathbb{R}$ *as follows:*

$$\mu(\gamma) = \frac{1}{2}, \quad \kappa(\mathfrak{y}) = 2\mathfrak{y}.$$

Now, for all $\varkappa, \mathfrak{y} \in U$ *with* $\sigma(\varkappa, \mathfrak{y}) > 0$ *and* $\mathfrak{y}_1, \mathfrak{y}_2 \in [0, 1]$*, we compute*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}_1), \Psi(\mathfrak{y}, \mathfrak{y}_2)) = \mathcal{H}_\sigma(3 + \mathfrak{y}_1, 3 + \mathfrak{y}_2) = |\mathfrak{y}_1 - \mathfrak{y}_2| \leq \mu(\sigma(\varkappa, \mathfrak{y}))\sigma(\varkappa, \mathfrak{y}),$$

demonstrating that condition (ii) is satisfied.
Similarly, for any $\varkappa \in U$ *with* $\sigma(\varkappa, \varkappa) = 0$ *and* $\mathfrak{y}, \mathfrak{z} \in [0, 1]$*, we have*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\varkappa, \mathfrak{z})) = |\mathfrak{y} - \mathfrak{z}| = \frac{1}{2}|\kappa(\mathfrak{y}) - \kappa(\mathfrak{z})|,$$

so condition (iii) is also fulfilled.
However, condition (iv) fails to hold. Indeed, suppose $\mathfrak{y} \in [0,1]$ *is such that*

$$\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) = \sigma(U, V),$$

for some $\varkappa \in U^*$. *Since* $U^* = \{1\}$, *this implies* $\mathfrak{y} = 0$, *yielding* $\Psi(1,0) = 3$. *But for any small* $\varepsilon_0 > 0$, *we observe that* $\Psi(\{1\}, \mathfrak{y}^*) = 3 + \mathfrak{y}^* \not\subseteq \{3\}$ *for* $\mathfrak{y}^* \in (0, \varepsilon_0)$. *Hence, no neighborhood of* $\mathfrak{y} = 0$ *is entirely mapped into* $V_0 = \{3\}$, *violating condition (iv).*
Thus, although $\Psi(\cdot, 0)$ *possesses a best proximity point in* U, *the mapping* $\Psi(\cdot, 1)$ *does not admit a best proximity point.*

Corollary 5.3.1. *Consider the partial metric space* $(\mathcal{W}, \sigma)$ *which is 0-complete and suppose* $U, V \subseteq \mathcal{W}$ *are its non-empty and closed subsets with* $U_0 \neq \varnothing$ *and* $U^* \subseteq U$ *is non-empty. Suppose the pair* (U, V) *possesses the weak P-property, and consider a mixed multivalued mapping* $\Psi : U \times [0,1] \to \mathcal{CB}_\sigma(V) \cup V$ *that satisfies:*

(i) $\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) > \sigma(U, V)$ *for every* $\varkappa \in U \setminus U^*$ *and* $\mathfrak{y} \in [0,1]$,

(ii) $\exists\ q \in [0,1)$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(w, z)) \leq q\sigma(\varkappa, w),$$

$\forall\ \varkappa, w \in U$ *with* $\sigma(\varkappa, w) > 0$ *and* $\mathfrak{y}, z \in [0,1]$,

(iii) $\forall\ \mathfrak{y}, \mathfrak{z} \in [0,1]$ *and* $\varkappa \in U$ *with* $\sigma(\varkappa, \varkappa) = 0$, $\exists$ *a function* $\kappa : [0,1] \to \mathbb{R}$ *which is continuous such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\varkappa, \mathfrak{z})) \leq q|\kappa(\mathfrak{y}) - \kappa(\mathfrak{z})|,$$

(iv) If $\mathfrak{y} \in [0,1]$ *satisfies* $\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) = \sigma(U, V)$ *for some* $\varkappa \in U^*$, *then* $\exists\ \varepsilon_\mathfrak{y} > 0$ *such that*

$$\Psi(U_0, \mathfrak{y}^*) \subseteq V_0, \quad \forall \mathfrak{y}^* \in (\mathfrak{y} - \varepsilon_\mathfrak{y}, \mathfrak{y} + \varepsilon_\mathfrak{y}),$$

(v) If $\sigma(\varkappa, \Psi(\varkappa, \mathfrak{y})) = \sigma(U, V)$, *then* $\Psi(\varkappa, \mathfrak{y})$ *has a single element.*

If the mapping $\Psi(\cdot, 0)$ *admits a best proximity point in* U, *then* $\Psi(\cdot, 1)$ *also possesses a* best proximity point *in* U.

Corollary 5.3.2. *Consider the partial metric space $(\mathcal{W}, \sigma)$ which is 0-complete and let $U \subseteq \mathcal{W}$ be a non-empty closed subset with $U^* \subseteq U$ non-empty. Suppose that $\Psi : U \times [0,1] \to \mathcal{W} \cup \mathcal{CB}_\sigma(\mathcal{W})$ is a mixed multivalued contraction fulfilling:*

(i) $\varkappa \notin \Psi(\varkappa, \mathfrak{y})\ \forall\ \varkappa \in U \setminus U^*$ *and* $\mathfrak{y} \in [0,1]$,

(ii) $\exists$ *a function* $\mu \in \mathcal{H}$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(w, z)) \leq \mu(\sigma(\varkappa, w)) \cdot \sigma(\varkappa, w),$$

$\forall\ \varkappa, w \in U$ *with* $\sigma(\varkappa, w) > 0$ *and* $\mathfrak{y}, z \in [0,1]$,

(iii) For each $\mathfrak{y}, \mathfrak{z} \in [0,1]$ *and* $\varkappa \in U$ *with* $\sigma(\varkappa, \varkappa) = 0$, $\exists$ *a continuous function* $\kappa : [0,1] \to \mathbb{R}$ *and constant* $q \in [0,1)$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\varkappa, \mathfrak{z})) \leq q|\kappa(\mathfrak{y}) - \kappa(\mathfrak{z})|,$$

(iv) If $\varkappa \in \Psi(\varkappa, \mathfrak{y})$ *for some* $\mathfrak{y} \in [0,1]$ *and* $\varkappa \in U^*$, $\exists\ \varepsilon_\mathfrak{y} > 0$ *such that*

$$\Psi(U, \mathfrak{y}^*) \subseteq U, \quad \forall \mathfrak{y}^* \in (\mathfrak{y} - \varepsilon_\mathfrak{y}, \mathfrak{y} + \varepsilon_\mathfrak{y}),$$

(v) If $\varkappa \in \Psi(\varkappa, \mathfrak{y})$, *then* $\Psi(\varkappa, \mathfrak{y}) = \{\varkappa\}$.

If $\Psi(\cdot, 0)$ *possesses a fixed point in* U, *then so does* $\Psi(\cdot, 1)$.

Proof. Suppose $\exists$ a point $\varkappa \in U$ such that $\varkappa \in \Psi(\varkappa, 0)$. By the assumption, the hypotheses (i), (ii), (iii), and (v) of Theorem 4.1 are fulfilled by the mapping Ψ.

Now, observe that if $U = \mathcal{W}$, then we have $U_0 = U$, and hence the condition (iv) is trivially met in this case. Therefore, all the hypothesis of Theorem 5.3.1 are fulfilled. Applying Theorem 5.3.1, we deduce that $\exists$ an element $\varkappa^* \in U$ such that

$$\sigma(\varkappa^*, \Psi(\varkappa^*, 1)) = \sigma(U, \mathcal{W}).$$

Furthermore, since the partial metric satisfies $\sigma(U, \mathcal{W}) \leq \sigma(\varkappa^*, \varkappa^*)$ and

$$\sigma(U, \mathcal{W}) \leq \sigma(\Psi(\varkappa^*, 1), \Psi(\varkappa^*, 1)).$$

We conclude that

$$\sigma(\varkappa^*, \Psi(\varkappa^*, 1)) = \sigma(\varkappa^*, \varkappa^*) = \sigma(\Psi(\varkappa^*, 1), \Psi(\varkappa^*, 1)).$$

Therefore $\varkappa^* \in \Psi(\varkappa^*, 1)$, and by the uniqueness condition (v), it follows that $\Psi(\varkappa^*, 1) = \{\varkappa^*\}$. Hence, $\varkappa^*$ is a fixed point of the mapping $\Psi(\cdot, 1)$. □

Corollary 5.3.3. *Consider the partial metric space $(\mathcal{W}, \sigma)$ which is 0-complete and let $U \subseteq \mathcal{W}$ be its subset which is non-empty and closed with $U^* \subseteq U$ non-empty. Let $\Psi : U \times [0,1] \to \mathcal{W} \cup \mathcal{CB}_\sigma(\mathcal{W})$ be a mixed multivalued contraction such that:*

(i) $\varkappa \notin \Psi(\varkappa, \mathfrak{y})$ *for all* $\varkappa \in U \setminus U^*$ *and* $\mathfrak{y} \in [0,1]$,

(ii) $\exists\ q \in [0,1)$ *such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\mathfrak{y}, z)) \leq q\sigma(\varkappa, \mathfrak{y}),$$

$\forall\ \varkappa, \mathfrak{y} \in U$ *with* $\sigma(\varkappa, \mathfrak{y}) > 0$ *and* $\mathfrak{y}, z \in [0,1]$,

(iii) $\forall\ \mathfrak{y}, \mathfrak{z} \in [0,1]$ *and* $\varkappa \in U$ *with* $\sigma(\varkappa, \varkappa) = 0$, $\exists$ *a function* $\kappa : [0,1] \to \mathbb{R}$ *which is continuous such that*

$$\mathcal{H}_\sigma(\Psi(\varkappa, \mathfrak{y}), \Psi(\varkappa, \mathfrak{z})) \leq q|\kappa(\mathfrak{y}) - \kappa(\mathfrak{z})|,$$

(iv) If $\varkappa \in \Psi(\varkappa, \mathfrak{y})$ *for some* $\varkappa \in U^*$, *then* $\exists\ \varepsilon_\mathfrak{y} > 0$ *such that*

$$\Psi(U, \mathfrak{y}^*) \subseteq U, \quad \forall \mathfrak{y}^* \in (\mathfrak{y} - \varepsilon_\mathfrak{y}, \mathfrak{y} + \varepsilon_\mathfrak{y}),$$

(v) If $\varkappa \in \Psi(\varkappa, \mathfrak{y})$, *then* $\Psi(\varkappa, \mathfrak{y}) = \{\varkappa\}$.

If $\Psi(\cdot, 0)$ *admits a fixed point in* U, *then* $\Psi(\cdot, 1)$ *also possesses a fixed point in* U.

Chapter 6

Conclusion and Future Work

Conclusions

Nadler's classical fixed point theorem, a foundational result in multivalued analysis, asserts that every multivalued contraction defined on a complete metric space possesses at least one fixed point. Over time, this theorem has inspired numerous generalizations and extensions, which are well documented in the literature. Notably, all these developments deal exclusively with self-mappings. In situations where the fixed point equation $\Gamma(\varkappa) = \varkappa$ lacks a solution, attention is directed toward identifying a point $\varkappa$ that remains close to its image under Γ, i.e., such that $\varkappa$ is in proximity to $\Gamma(\varkappa)$.

In this thesis, we studied the theory of best proximity points for multivalued non-self mappings in metric and partial metric spaces. Inspired by the fact that classical fixed point equations of the form $\Gamma\varkappa = \varkappa$ may not always have a solution especially when the mapping is not self-mapping we turned our attention to the study of approximate solutions, namely best proximity points. These are points $\varkappa \in U \subset \mathcal{Y}$ for which the distance to their image under the mapping $\Gamma : U \to V$ attains the minimal possible value, $\mathrm{dist}(U, V)$.

We began our study with foundational results from fixed point theory, starting with Banach contraction principle and progressing to more general theorems such as Nadler's and Feng-Liu results for multivalued mappings. Recognizing the limitations of fixed point theory in non-self mapping contexts, we focused on establishing best proximity point theorems that extend and generalize existing frameworks. Significant attention was given to various types of multivalued contraction mappings, including cyclic contractions, Feng-Liu type contractions, and F-contractions.

In these studies, we adopted weaker assumptions such as the *weak P-property*, in place of the classical P-property, and required that the image $\Gamma(\varkappa)$ be compact for each $\varkappa \in U$. This enabled the extension of best proximity point results to more general and practical settings.

Future work

Inspired by the control function technique introduced by Wardowski [29], and building on the subsequent work of Altun et al. [7] and Debnath [11], we will study some best proximity point results for almost F-contractions. These mappings offer greater flexibility, although they may not guarantee the continuity of the control function F.

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